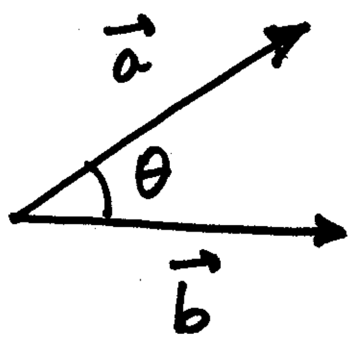
Outline

1. Angles and orthogonality.
2. Vector projection.
3. Cross product.
4. Parallelpipeds.

1. Angles and Orthogonality



$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|}.$$

- Two vectors $\vec{a}$ and $\vec{b}$ are said to be <u>orthogonal</u> iff:

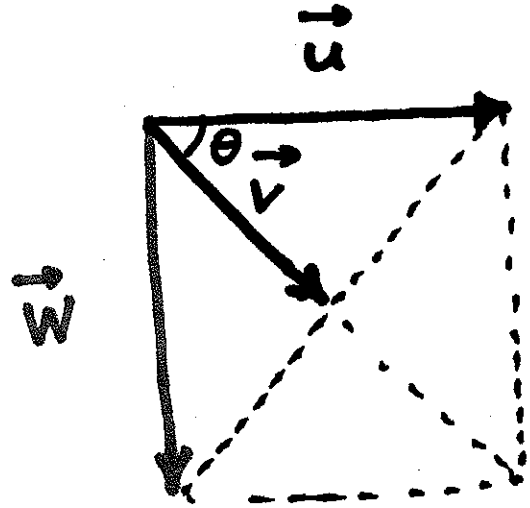
$$\vec{a} \bullet \vec{b} = 0.$$

- In 2D and 3D, if two vectors are <u>orthogonal</u>, then geometrically they are <u>perpendicular</u>.

<u>Example</u>

$\vec{u}$ = unit vector (i.e. $|\vec{u}| = 1$)

Find: $\vec{u} \bullet \vec{w}$ and $\vec{u} \bullet \vec{v}$ given:



(A square)

Solution

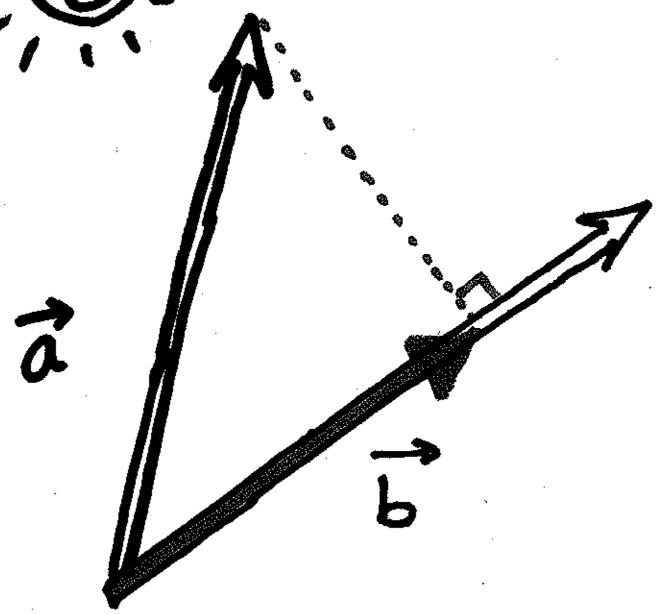
(a) $\vec{u} \bullet \vec{w} = 0$ ($\vec{u}$ and $\vec{w}$ are perpendicular).

(b). $\vec{u} \bullet \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos(\theta)$

$\theta = \frac{\pi}{4} \qquad |\vec{u}| = 1 \qquad |\vec{v}| = \frac{\sqrt{2}}{2}$

$\vec{u} \bullet \vec{v} = (1)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{2}{4} = \frac{1}{2}.$

2. Vector Projection

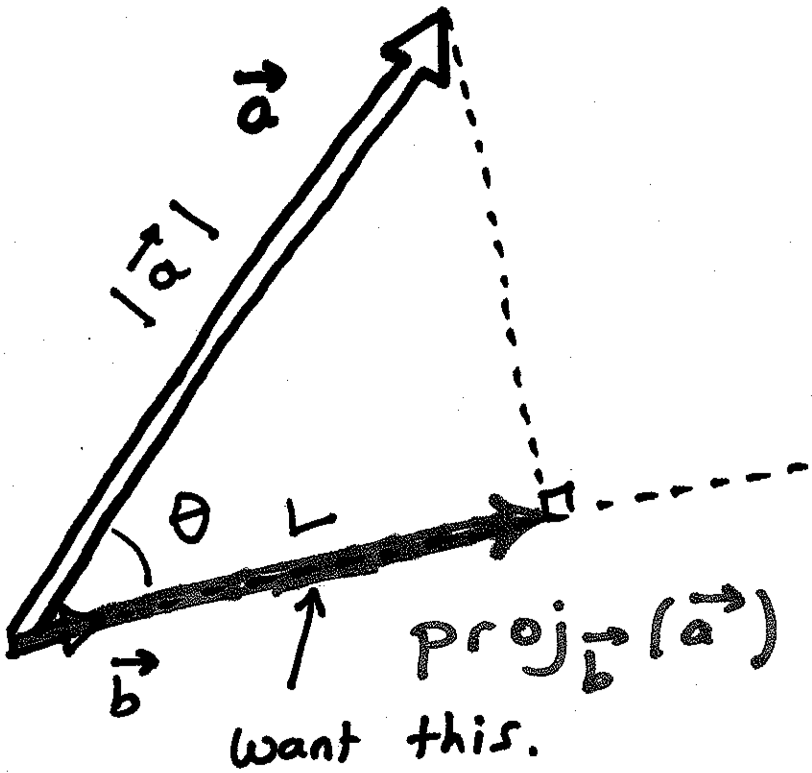


The projection of vector $\vec{a}$ onto vector $\vec{b}$ gives a new vector, usually called: $\text{proj}_{\vec{b}}(\vec{a})$.

- $\text{proj}_{\vec{b}}(\vec{a})$ is a <u>vector</u>

- $$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|^2}\,\vec{b}$$

- $\text{proj}_{\vec{b}}(\vec{a})$ is in the same direction (or the opposite direction) as $\vec{b}$.



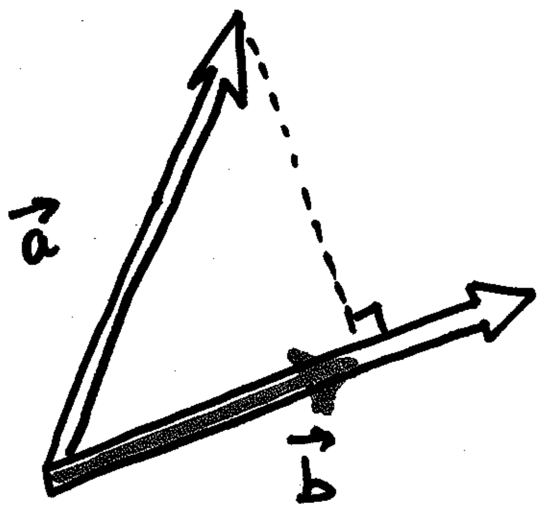
$$\frac{L}{|\vec{a}|} = \cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$L = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}$$

$$\mathrm{proj}_{\vec{b}}(\vec{a}) = \frac{1}{|\vec{b}|} \cdot \vec{b} \cdot \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}$$

$$= \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|^2} \vec{b}$$

1. Vector Projection



$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|^2}\,\vec{b}$$

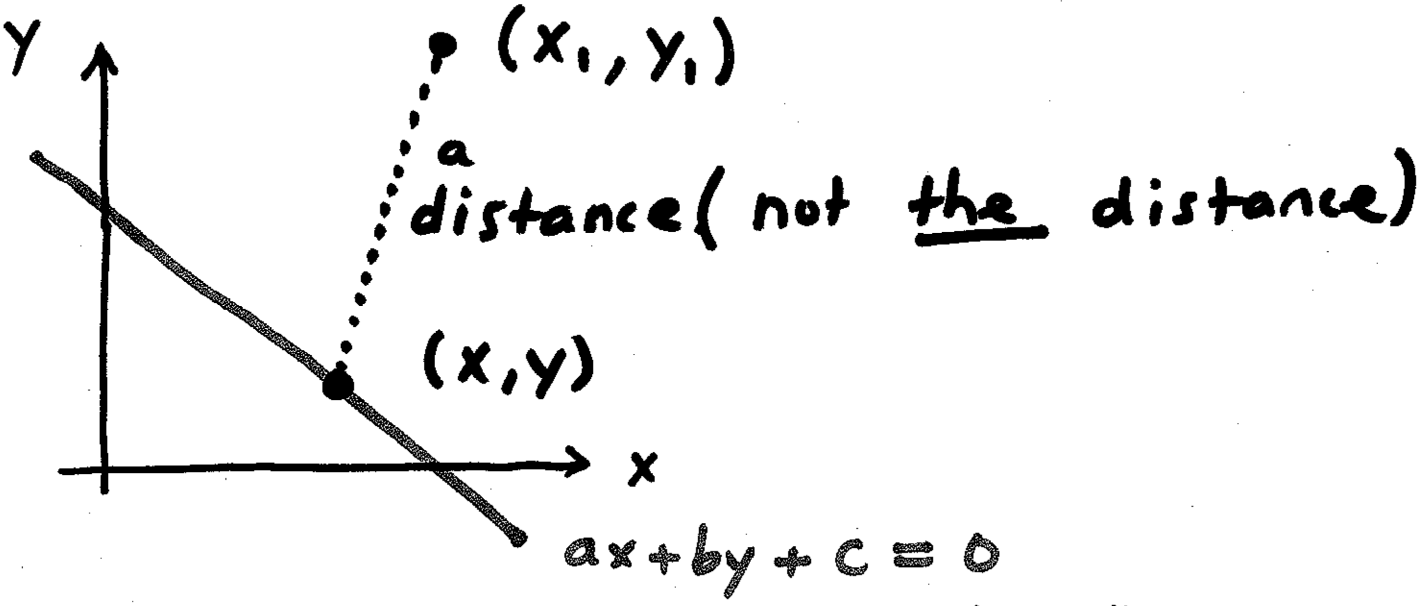
- $\text{proj}_{\vec{b}}(\vec{a})$ is a vector that is parallel to $\vec{b}$.

Example

Suppose $ax + by + c = 0$ $(a \neq 0, b \neq 0)$ is a line in the x-y plane and (x_1, y_1) is a point that's not on the line.

Distance from point to line $= \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Proof (Calculus)



Distance we want is "d." We can get d by finding the minimum value of:

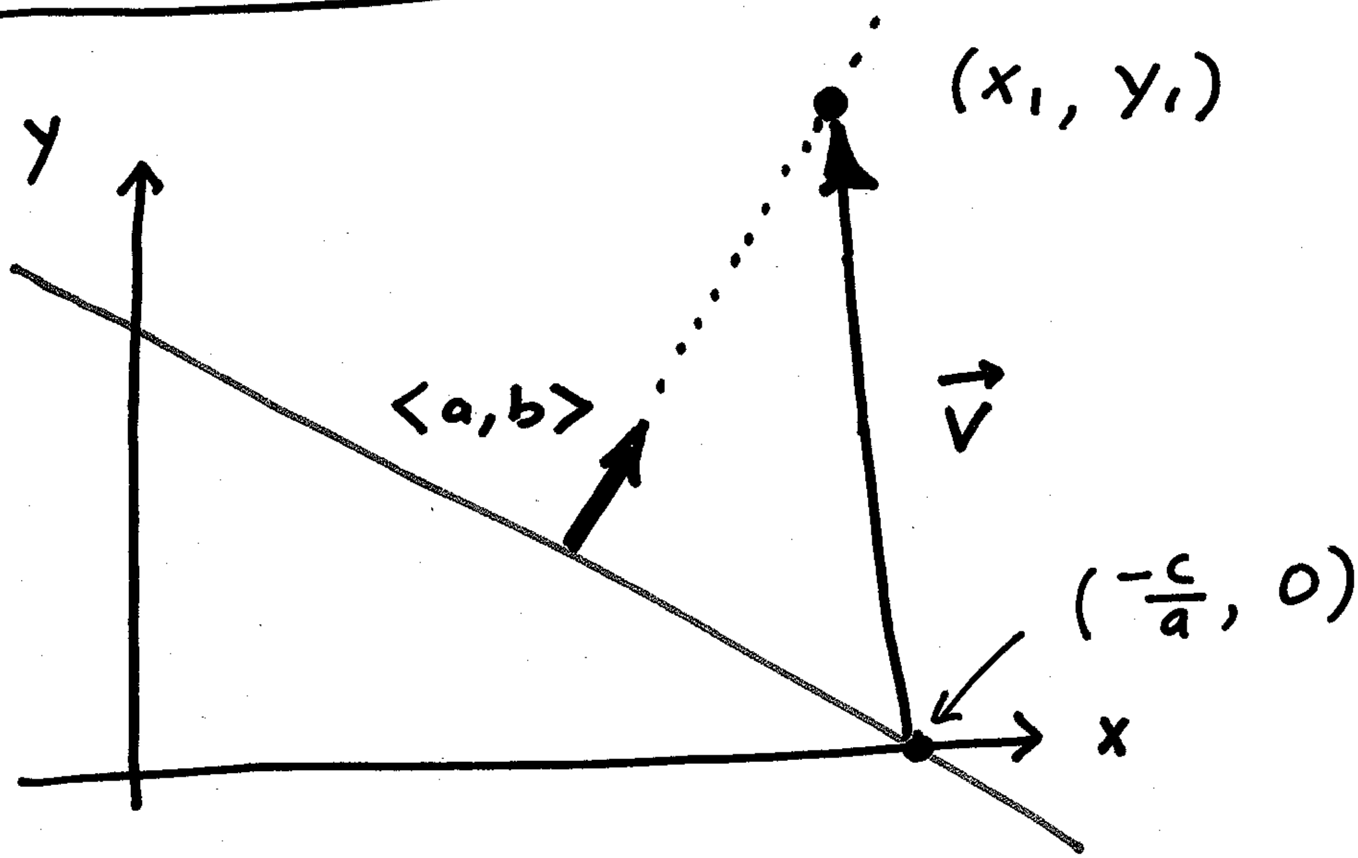
$$\sqrt{(x-x_1)^2+(y-y_1)^2} \qquad y = -\frac{a}{b}x - \frac{c}{b}$$

$$= \sqrt{(x-x_1)^2+\left(\frac{-a}{b}x - \frac{c}{b} - y_1\right)^2}$$

Idea: Take deriv,
Set deriv $= 0$,
Solve for x,
Plug this value of x back in.

Not much fun to actually do.

Proof (Vector Projection)



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$y = -\frac{a}{b}x - \frac{c}{b}$

Vector parallel to line $= \langle b, -a \rangle$

Vector perpendicular to line $= \langle a, b \rangle$

$$\vec{v} = \langle x_1 - -\frac{c}{a}, \; y_1 - 0 \rangle = \langle x_1 + \frac{c}{a}, \; y_1 \rangle$$

$$\mathrm{proj}_{\langle a,b\rangle}(\vec{v}) = \frac{\vec{v} \bullet \langle a,b\rangle}{|\langle a,b\rangle|^2} \cdot \langle a,b\rangle$$

$$= \frac{ax_1 + c + by_1}{a^2 + b^2} \langle a, b\rangle$$

The distance from the point (x_1, y_1) to the line is the length of $\mathrm{proj}_{\langle a,b\rangle}(\vec{v})$.

$$|\mathrm{proj}_{\langle a,b\rangle}(\vec{v})| = \frac{|ax_1 + by_1 + c|}{a^2 + b^2}\sqrt{a^2+b^2}$$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \quad \blacksquare$$

2. Cross Product

- Cross product is only defined for 3D vectors.

- $\vec{a} = \langle a_1, a_2, a_3 \rangle$
  $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2 \, , a_3 b_1 - a_1 b_3 , a_1 b_2 - a_2 b_1 \rangle$$

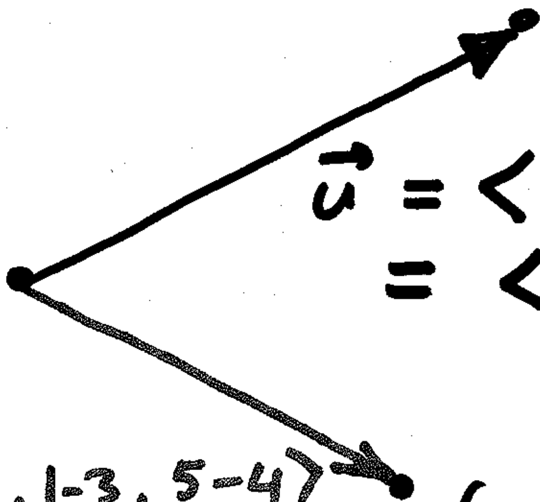
- Result, $\vec{a} \times \vec{b}$, is a vector that is perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$.

Example

Find an equation for the plane in $\mathbb{R}^3$ that contains the points:

$(2, 1, 5)$ $(-1, 3, 4)$ $(3, 0, 6)$

Solution



$$\vec{u} = \langle 3-2, 0-1, 6-5 \rangle$$
$$= \langle 1, -1, 1 \rangle$$

$$\vec{v} = \langle 2 - -1, 1-3, 5-4 \rangle$$
$$= \langle 3, -2, 1 \rangle$$

Let $\vec{n}$ be a normal vector to the plane.

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= \begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & -1 & 1 & 1 & -1 \\ 3 & -2 & 1 & 3 & -2 \end{array}$$

$$= \langle (-1)(1) - (1)(-2), (1)(3) - (1)(1), (1)(-2) - (-1)(3) \rangle$$

$$= \langle 1, 2, 1 \rangle$$

Equation of plane:

$$1 \cdot (x-2) + 2 \cdot (y-1) + 1 \cdot (z-5) = 0.$$

Angle formula for Cross Product

Let $\vec{a}, \vec{b}$ be 3D vectors.

Then:

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Note: If $\vec{a}$ and $\vec{b}$ are parallel (i.e. $\theta = 0$ or $\theta = \pi$) then:

$$\vec{a} \times \vec{b} = \underset{\langle 0,\, 0,\, 0 \rangle.}{\underset{\uparrow}{\vec{0}}}.$$