Solutions to Homework #9

Problems from Pages 658-659 (Section 11.8)

10. Given $f(x, y, z) = x^4 + y^4 + z^4$ and the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$, the three equations that we get by setting up the Lagrange multiplier equations $\nabla f = \lambda \nabla g$:

$$4x^3 = 2\lambda x \qquad 4y^3 = 2\lambda y \qquad 4z^3 = 2\lambda z.$$

We will break the solution of these three equations into three cases.

Case 1: *x*, *y* and *z* are all nonzero.

In this case, we can simplify the system of three equations to get:

$$x^2 = y^2 = z^2 = \lambda/2.$$

Substituting this into $g(x, y, z) = 0$ and solving gives that $x^2 = y^2 = z^2 = 1/3$. This means that there are a total of eight points that have x, y, and z all non-zero, each of which looks like:

$$(x,y,z) = \left(\pm\tfrac{1}{\sqrt{3}}, \pm\tfrac{1}{\sqrt{3}}, \pm\tfrac{1}{\sqrt{3}}\right).$$

At each such point, $f(x, y, z) = 1/3$.

Case 2: Exactly one variable (*x*, *y* or *z*) is equal to zero.

For the sake of demonstration, let us suppose that $x = 0$ and that neither y nor z is zero. Then the Lagrange multiplier equations give that:

$$y^2 = z^2 = \lambda/2.$$

Substituting these values into the constraint equation $g(x, y, z) = 0$ and solve for y and z gives a collection of four points, each of which has the form:

$$(x,y,z) = \left(0, \pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}\right).$$

There are two additional families of points like this, one corresponding to $y = 0$ and the other corresponding to $z = 0$. At all of the points in all of the three families identified here, $f(x, y, z) = ½$.

Case 3: Exactly two variables are equal to zero.

For the sake of demonstration, let us suppose that $x = 0$ and that $y = 0$, and that z is not zero. Substituting these values into the constraint $g(x, y, z) = 0$ give us the solution $z = \pm 1$. At each solution point, $f(x, y, z) = 1$.

Solution of problem: The maximum value of $f(x, y, z)$ is 1 and the minimum value of $f(x, y, z)$ is 1/3.

12. The Lagrange multiplier equations for the function:

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$$

subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1 = 0$$

are all of the form:

$$1 = 2\lambda x_j$$

with $j = 1, \ldots, n$. If $\lambda = 0$, then none of the Lagrange multiplier equations can be solved, so we will assume that λ is not equal to zero. This gives that:

$$x_1 = x_2 = \ldots = x_n.$$

Substituting the fact that all of the variables must be equal at the solution points into the constraint $g(x_1, \ldots, x_n) = 0$ gives that for each $j = 1, \ldots n$:

$$x_j = \frac{\pm 1}{\sqrt{n}}.$$

Then at each solution point, the value of the function $f(x_1, \ldots, x_n)$ will be a sum of n copies of the x_j given above, with some positive and some negative.

The maximum value of $f(x_1, \ldots, x_n)$ will be attained when all of the terms added together are positive, and will produce a maximum value of $f(x_1, \ldots, x_n)$ equal to $\sqrt{n}$.

The minimum value of $f(x_1, \ldots, x_n)$ will be attained when all of the terms added together are negative, and will produce a minimum value of $f(x_1, \ldots, x_n)$ equal to $-\sqrt{n}$.

22. In this problem we are asked to minimize the cost function:

$$C = mL + nK,$$

where L and K are subject to the constraint $bL^{\alpha}K^{1-\alpha} = Q$. This calculation is one of the most beautiful applications of the Lagrange Multipliers technique in all of economics.

We will write the functions involved in the familiar Lagrange Multiplier notation as:

$$f(L, K) = mL + nK$$

$$g(L, K) = bL^{\alpha}K^{1-\alpha} - Q.$$

The two gradients that we need to calculate are as follows:

$$\nabla f = < m, n >$$

$$\nabla g = < \alpha b L^{\alpha-1}K^{1-\alpha}, (1-\alpha) bL^{\alpha}K^{-\alpha} >.$$

The Lagrange multiplier equations, $\nabla f = \lambda \nabla g$, are:

$$m = \lambda \alpha b L^{\alpha-1}K^{1-\alpha}$$

$$n = \lambda(1-\alpha) bL^{\alpha}K^{-\alpha}.$$

Multiplying the first equation by L and the second equation by K and making the substitution $bL^{\alpha}K^{1-\alpha} = Q$ into both gives:

$$mL = \lambda \alpha Q$$

$$nK = \lambda(1-\alpha)Q.$$

Solving both equations for λQ and equating allows us to solve for L in terms of K to get:

$$L = \frac{n\alpha}{m(1-\alpha)}K.$$

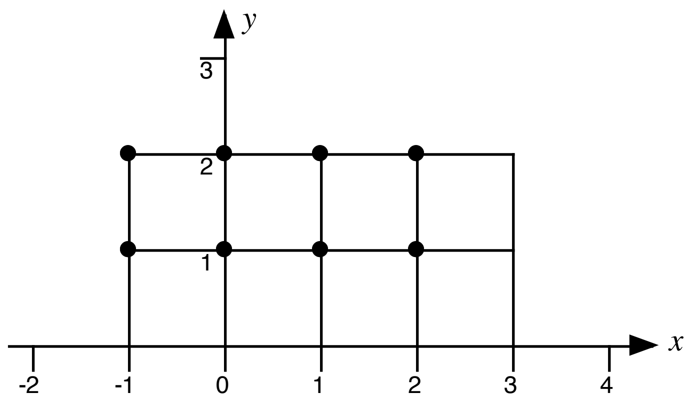
We will now use this equation to substitute for L in the constraint equation $g(L, K) = 0$ and solve for K. Doing this gives:

$$K = \frac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}} \qquad \text{and} \qquad L = \frac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}.$$

Who knew the dismal science of economics could be so pretty!

Problems from Pages 672-674 (Section 12.1)

2. The region $[-1, 3]\times[0, 2]$ is divided up into eight squares each with width $\Delta x = 1$ and height $\Delta y = 1$ as shown in the diagram (below).



Each of the little squares is the base of a French fry. To get the height of each French fry in this problem, we have to evaluate the function at the top left-hand corner of each square (indicated by the dots in the diagram given above).

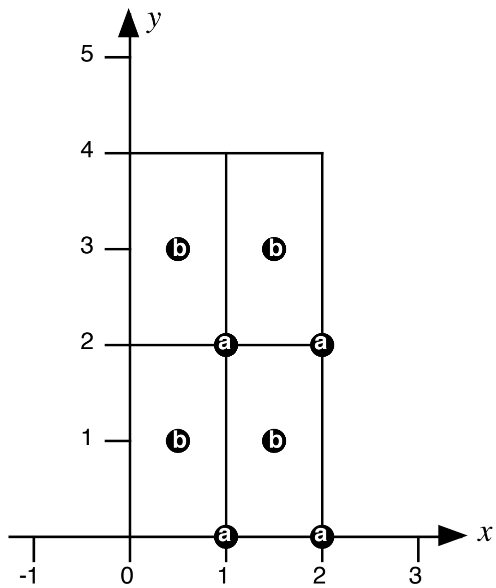
Doing this gives that the volume is approximated by:

$$\begin{aligned} \textit{Volume} \quad &\approx \quad f(-1, 1)\cdot\Delta x\cdot\Delta y + f(-1, 2)\cdot\Delta x\cdot\Delta y + f(0, 1)\cdot\Delta x\cdot\Delta y + \\ &\quad\quad f(0, 2)\cdot\Delta x\cdot\Delta y + f(1, 1)\cdot\Delta x\cdot\Delta y + f(1, 2)\cdot\Delta x\cdot\Delta y + \\ &\quad\quad f(2, 1)\cdot\Delta x\cdot\Delta y + f(2, 2)\cdot\Delta x\cdot\Delta y \\ &= \quad -1 + 2 + 1 + 4 - 1 + 2 - 7 - 4 \\ &= \quad -4. \end{aligned}$$

4. In this problem the heights of the French fries are given by the function:

$$f(x, y) = x + 2y^2.$$

The region $[0, 2]\times[0, 4]$ of the xy-plane is divided into four smaller rectangles with width $\Delta x = 1$ and height $\Delta y = 2$ as shown in the diagram (below). These smaller rectangles form the bases of the French fries. The dots on the diagram indicate where the function $f(x, y)$ is evaluated in Part (a) and Part (b) of this problem.



(a) In this case, the function is evaluated at the bottom, right-hand corner of each rectangle.

Volume $\approx$ $f(1, 0)\cdot\Delta x\cdot\Delta y + f(1, 2)\cdot\Delta x\cdot\Delta y + f(2, 0)\cdot\Delta x\cdot\Delta y + f(2, 2)\cdot\Delta x\cdot\Delta y$

$= 2(1 + 9 + 2 + 10)$

$= 44.$

(b) In this case, the function is evaluated at the center of each rectangle.

Volume $\approx$ $f(0.5, 1)\cdot\Delta x\cdot\Delta y + f(0.5, 3)\cdot\Delta x\cdot\Delta y + f(1.5, 1)\cdot\Delta x\cdot\Delta y + f(1.5, 3)\cdot\Delta x\cdot\Delta y$

$= 2(2.5 + 18.5 + 3.5 + 19.5)$

$= 88.$

(c) The double integral that gives the exact value of the volume is:

$$\text{Volume} = \int_0^4 \int_0^2 \left(x + 2y^2\right) dx\,dy$$

$$= \int_0^4 \left[\tfrac{1}{2}x^2 + 2xy^2\right]_0^2 dy$$

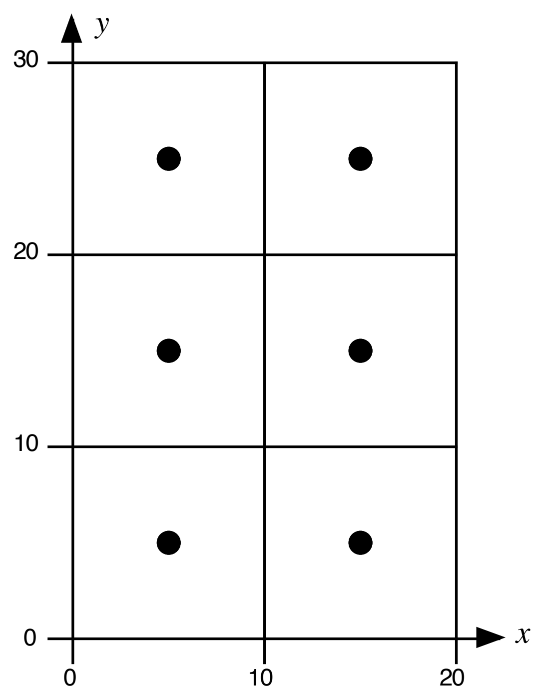
$$= \int_0^4 \left(2 + 4y^2\right) dy$$

$$= \left[2y + \tfrac{4}{3}y^3\right]_0^4$$

$$= 280/3.$$

6. There are many, many ways to set up and calculate an estimate for the volume of the pool, given the information supplied in this problem. The answer or estimate that you obtained for the volume of the pool may be a little different (and maybe a little more accurate) than the one given here, but your number should be reasonably close to the number listed below.

Let $f(x, y)$ give the depth of the pool (in meters) x meters to the right and y meters above the lower left hand corner of the pool (which corresponds to the depth 2 in the upper left hand corner of the table given on page 673). We will break the volume of the pool up into French fries with bases that have width $\Delta x = 10$ and height $\Delta y = 10$, as shown in the diagram shown below. (The dots are the points where the function is evaluated to obtain the height of each French fry.)



Then the volume, V, of the pool will be approximated by:

$$V \approx f(5, 5)\cdot\Delta x\cdot\Delta y + f(5, 15)\cdot\Delta x\cdot\Delta y + f(5, 25)\cdot\Delta x\cdot\Delta y +$$

$$f(15,5)\cdot\Delta x\cdot\Delta y + f(15,15)\cdot\Delta x\cdot\Delta y + f(15,25)\cdot\Delta x\cdot\Delta y$$

$$= 100(3+7+10+3+5+8)$$

$$= 3600 \text{ cubic feet.}$$

12. $$\int_2^4\int_{-1}^1 \left(x^2+y^2\right)dydx = \int_2^4\left[x^2y+\tfrac{1}{3}y^3\right]_{-1}^1 dx = \int_2^4\left(2x^2+\tfrac{2}{3}\right)dx = \left[\tfrac{2}{3}x^3+\tfrac{2}{3}x\right]_2^4 = \tfrac{116}{3}.$$

32. The shadow cast in the xy-plane by this volume is the rectangle $[0, 3]\times[0, 2]$. The volume below the surface $z = 1 + (x-1)^2 + 4y^2$ over this rectangle is given by the double integral:

$$\int_0^3\int_0^2\left(1+(x-1)^2+4y^2\right)dydx = 44.$$

Problems from Pages 680-682 (Section 12.2)

8. The double integral is given by:

$$\int_1^2\int_0^{2x}\frac{4y}{x^3+2}dydx = \int_1^2\left[\frac{2y^2}{x^3+2}\right]_0^{2x}dx = \int_1^2\frac{8x^2}{x^3+2}dx = \left[\tfrac{8}{3}\ln\left(\left|x^3+2\right|\right)\right]_1^2 = \tfrac{8}{3}\ln\left(\tfrac{10}{3}\right).$$

12. The double integral is given by:

$$\int_0^1\int_{x^2}^{\sqrt{x}}(x+y)dydx = \int_0^1\left[xy+\tfrac{1}{2}y^2\right]_{x^2}^{\sqrt{x}}dx = \int_0^1\left(x^{3/2}+\tfrac{1}{2}x-x^3-\tfrac{1}{2}x^4\right)dx = \left[\tfrac{2}{5}x^{5/2}+\tfrac{1}{4}x^2-\tfrac{1}{4}x^4-\tfrac{1}{10}x^5\right]_0^1 = \tfrac{3}{10}$$