Solutions to Homework #8

Problems from Pages 631-633 (Section 11.5)

18. Given $u = (r^2 + s^2)^{1/2}$, $r = y + x\cdot\cos(t)$ and $s = x + y\cdot\sin(t)$, the Chain rule for partial derivatives gives that:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{r\cdot\cos(t) + s}{\sqrt{r^2 + s^2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{r + s\cdot\sin(t)}{\sqrt{r^2 + s^2}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \frac{-r\cdot x\cdot\sin(t) + s\cdot y\cdot\cos(t)}{\sqrt{r^2 + s^2}}.$$

30. **(a)** Since $\partial W/\partial T$ is negative, a rise in average temperature (with rainfall remaining constant) will lead to a decrease in wheat production. Since $\partial W/\partial R$ is positive, an increase in rainfall (with temperature remaining constant) will lead to an increase in wheat production.

(b) We are given that $dT/dt = 0.15$ ºC per year and that $dR/dt = -0.1$ cm per year. Using the Chain rule for partial derivatives:

$$\frac{dW}{dt} = \frac{\partial W}{\partial T}\frac{dT}{dt} + \frac{\partial W}{\partial R}\frac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1.$$

So, wheat production will decrease by approximately 1.1 production units per year.

32. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. We are given that $dr/dt = 1.8$ inches per second and that $dh/dt = -2.5$ inches per second. Using the Chain rule for partial derivatives, the rate of change of the volume when $r = 120$ and $h = 140$ is:

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi(120)(140)}{3}(1.8) + \frac{\pi(120)^2}{3}(-2.5) = 8160\pi$$

cubic inches per second.

Problems from Pages 642-644 (Section 11.6)

24. **(a)** The gradient vector of $T(x,y,z) = 200e^{-x^2-3y^2-9z^2}$ evaluated at the point (2, −1, 2) is:

$$\nabla T(2,-1,2) = -400e^{-43} < 2,-3,18 >.$$

The vector that stretches from (2, −1, 2) to (3, −3, 3) is <1, −2, 1>. The unit vector that is parallel to this is:

$$\vec{u} = \frac{1}{\sqrt{6}} < 1,-2,1 >.$$

The directional derivative is:

$$D_{\vec{u}}T(1,-2,1) = \nabla T(2,-1,2) \bullet \vec{u} = \frac{-5200\sqrt{6}}{3e^{43}} \text{ °C per meter.}$$

(b) The direction that maximizes the directional derivative is:

$$\nabla T(2,-1,2) = -400e^{-43} < 2,-3,18 >.$$

(c) The maximum rate of increase from the point (2, −1, 2) is:

$$|\nabla T(2,-1,2)| = 400e^{-43}\sqrt{337} \text{ °C per meter.}$$

26. **(a)** The gradient vector of $z = f(x, y) = 1000 - 0.005x^2 - 0.01y^2$ is:

$$\nabla f = < -0.01x, -0.02y >.$$

The direction "due south" corresponds to the direction vector <0, −1>. The directional derivative at the point (*x*, *y*) = (60, 40) is then:

$$D_{<0,-1>}f(60,40) = < -0.6,-0.8 > \bullet < 0,-1 > = 0.8.$$

This means that if you walk due south, you will ascend at a rate of approximately 0.8 meters of altitude for every meter of horizontal distance than you cover.

(b) The direction of "northwest" corresponds to the direction vector $< \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$. The directional derivative at the point (*x*, *y*) = (60, 40) is then:

$$D_{\frac{1}{\sqrt{2}}<-1,1>}f(60,40) = < -0.6,-0.8 > \bullet < \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} > \approx -0.14.$$

This means that if you walk northwest, you will descend at a rate of approximately 0.14 meters of altitude for every meter of horizontal distance than you cover.

(c) The direction of the largest slope is $\nabla f(60,40) = < -0.6, -0.8 >$. The slope in this direction will be $|\nabla f(60,40)| = 1$, and the angle with the horizontal will be $\tan^{-1}(1) = \pi/4$ radians.

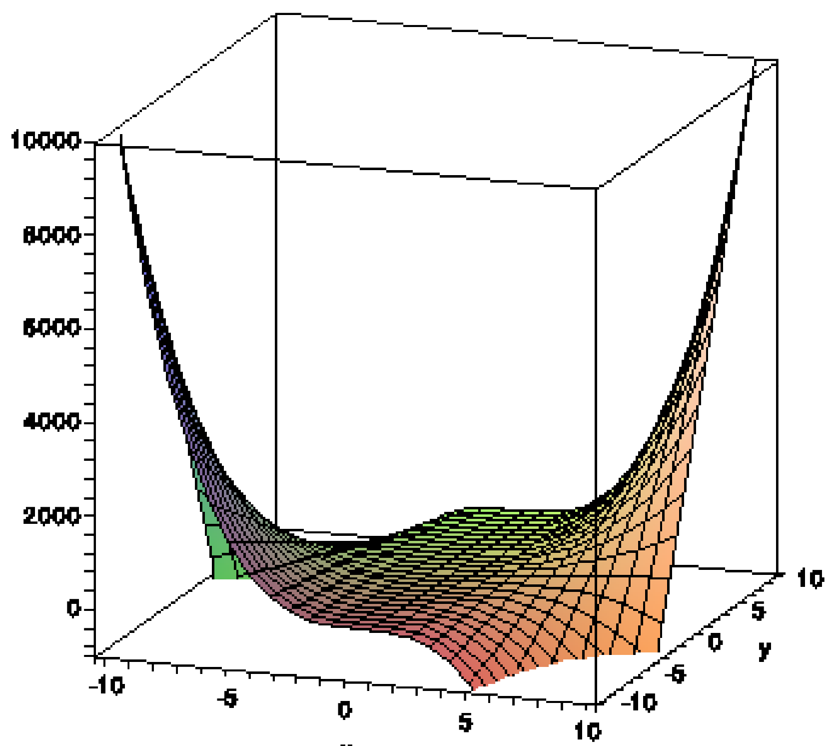
Problems from Pages 650-652 (Section 11.7)

4. To find the critical points of $f(x, y) = x^3y + 12x^2 - 8y$, we will find the solutions of the following equations:

$$\frac{\partial f}{\partial x} = 3x^2y + 24x = 0$$

$$\frac{\partial f}{\partial y} = x^3 - 8 = 0.$$

The only solution is $(x, y) = (2, -4)$. Evaluating the Jacobian determinant at this point gives $D = (f_{xx})(f_{yy}) - (f_{xy})^2 = -144 < 0$. Therefore, the critical point is a saddle point, which is confirmed by the graph shown below.



18. To find the critical points of $f(x, y) = \sin(x) + \sin(y) + \cos(x + y)$ we will find the solutions of the following equations:

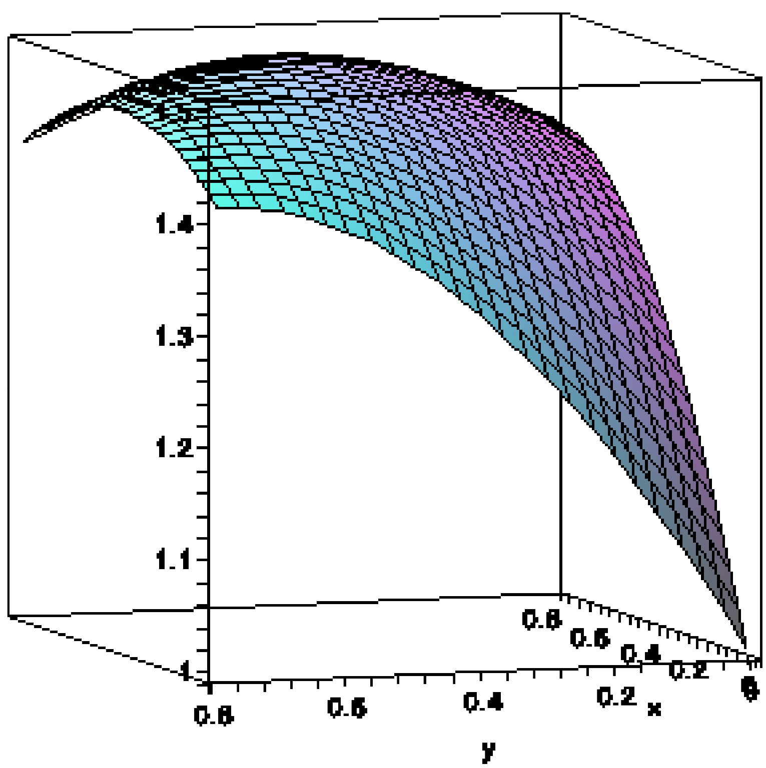
$$\frac{\partial f}{\partial x} = \cos(x) - \sin(x + y) = 0$$

$$\frac{\partial f}{\partial y} = \cos(y) - \sin(x + y) = 0.$$

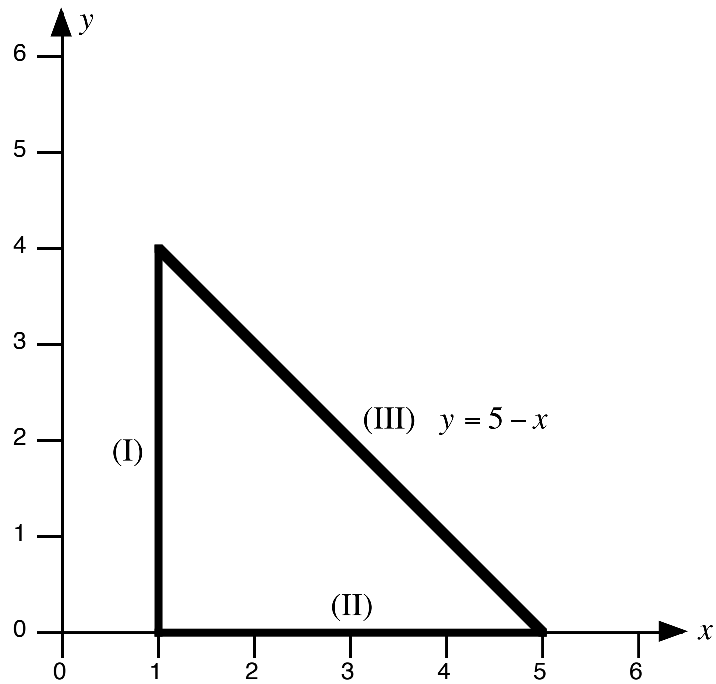
Subtracting one equation from the other gives that $\cos(x) = \cos(y)$. Because the values of x and y that we are interested in are restricted to $0 \le x \le \pi/4$ and $0 \le y \le \pi/4$, this implies that $x = y$. Using this, the equation that we need to solve becomes:

$$\cos(x) - \sin(2x) = \cos(x)(1 - 2\sin(x)) = 0.$$

The only solution of this equation that lies within $0 \le x \le \pi/4$ and $0 \le y \le \pi/4$ is $(x, y) = (\pi/6, \pi/6)$. Evaluating the Jacobian determinant at this point gives $D = (f_{xx})(f_{yy}) - (f_{xy})^2 = 0.75 > 0$. Evaluating $f_{xx}(\pi/6, \pi/6)$ gives -1 so the critical point at $(\pi/6, \pi/6)$ is a local maximum, as illustrated in the graph given below.



24. We will begin with a sketch of the region of interest.



We will begin by finding any critical points of $f(x, y) = 3 + xy - x - 2y$ that lie inside the region. To do this we will find the solutions of the following equations:

$$\frac{\partial f}{\partial x} = y - 1 = 0$$

$$\frac{\partial f}{\partial y} = x - 2 = 0.$$

The only solution is (2, 1), which does lie inside the region.

Next, we will find the points on the boundary of the region where the ordinary derivatives are equal to zero.

On (I): $x = 1$ and $f(1, y) = 2 - y = f_{(I)}(y)$. The derivative of this function is never equal to zero.

On (II): $y = 0$ and $f(x, 0) = 3 - x = f_{(II)}(x)$. The derivative of this function is never equal to zero.

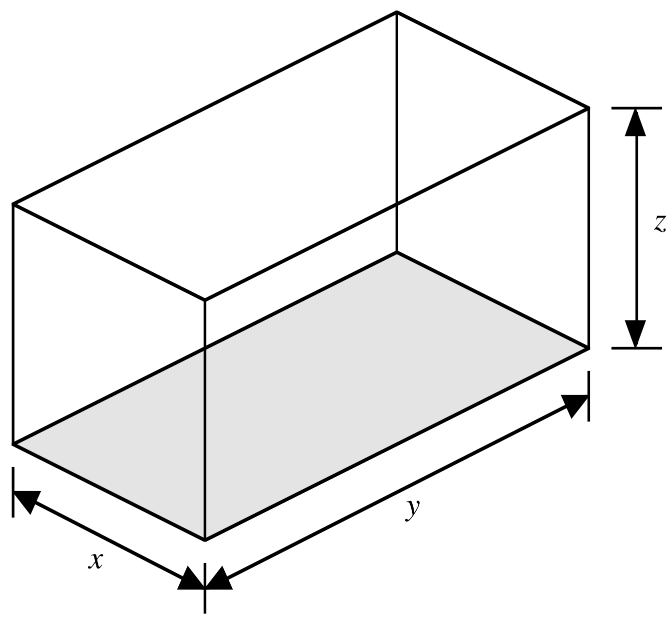
On (III): $y = 5 - x$ and $f(x, 5 - x) = -(x - 3)^2 + 2$. The derivative of this function is equal to zero at the point $(x, y) = (3, 2)$.

The only remaining points where the global maximum and global minimum could occur are at the corners of the region, (1, 0), (1, 4) and (5, 0).

To find the global minimum and global maximum of $f(x, y)$ we will evaluate the function at each of the points we have found.

x	y	f(x,y)	Comments
2	1	1	
3	2	2	Global maximum
1	0	2	Global maximum
1	4	–2	Global minimum
5	0	–2	Global minimum

42. We will begin with a sketch of the aquarium, showing the dimensions that we will use to set up this problem.



With the sides of aquarium labeled thus, the cost of the materials will be:

$$C = 5xy + 2xz + 2yz.$$

The constraint in this problem is provided by the volume, $xyz = V$, so that:

$$z = \frac{V}{xy},$$

and,

$$C = 5xy + \frac{2V}{x} + \frac{2V}{y}.$$

To find the critical point(s) we will set the partial derivatives of C equal to zero, and then solve to find x and y. Doing this gives that the only critical points occurs at:

$$x = y = \left(\frac{2V}{5}\right)^{\frac{1}{3}}.$$

This then gives $z = \left(\frac{25V}{4}\right)^{\frac{1}{3}}$. To demonstrate that this minimizes cost, note that for the values of x and y given above,

$$D = (C_{xx})(C_{yy}) - (C_{xy})^2 = \frac{16V^2}{\frac{4V^2}{25}} - 25 > 0$$

$$C_{xx} = 10 > 0,$$

confirming that the critical point detected is, in fact, a local minimum.

Problems from Pages 658-659 (Section 11.8)

8. The gradient vectors of f and g are as follows:

$$\nabla f = < 2xy^2z^2, 2x^2yz^2, 2x^2y^2z >$$

$$\nabla g = < 2x, 2y, 2z >.$$

The system of equations $\nabla f = \lambda \cdot \nabla g$ can be written out as the following set of three equations (where factors of 2 have been cancelled out on both sides).

$$x \cdot y^2 \cdot z^2 = \lambda \cdot x$$

$$x^2 \cdot y \cdot z^2 = \lambda \cdot y$$

$$x^2 \cdot y^2 \cdot z = \lambda \cdot z.$$

There are many ways that you can attack these equations. One way is to break the analysis into the two cases $\lambda = 0$ and $\lambda \neq 0$.

Case 1: $\boldsymbol{\lambda = 0}$

If $\lambda = 0$, then the product $xyz = 0$. To guarantee that $g(x, y, z) = 0$, and that $xyz = 0$, one or two (but not all three) of x, y or z may be zero. If two of the three coordinates are zero, then the remaining non-zero coordinate will equal 1 or −1. If only one of the three coordinates is equal to zero then the remaining two coordinates may take any values on a circle of radius 1.

At all of the points described above, $xyz = 0$ so that $f(x, y, z) = (xyz)^2 = 0$.

Case 2: $\boldsymbol{\lambda \neq 0}$

We can multiply the first of the three Lagrange multiplier equations by x, the second by y and the third by z to obtain:

$$\lambda \cdot x^2 = \lambda \cdot y^2 = \lambda \cdot z^2 = x^2 \cdot y^2 \cdot z^2.$$

If $\lambda \neq 0$ then we can cancel the λ's and conclude that $x^2 = y^2 = z^2$. Substituting this into the constraint equation $g(x, y, z) = 0$ gives that $x = \pm 1/\sqrt{3}$, $y = \pm 1/\sqrt{3}$ and $z = \pm 1/\sqrt{3}$. There are a total of eight points that all of the different combinations of these values yield. At each of the eight points, $f(x, y, z) = 1/27$.

The global minimum value of $f(x, y, z)$ is zero and the global maximum value of $f(x, y, z)$ is 1/27.