Solutions to Homework #5

Problems from Pages 543-545 (Section 10.4)

10. $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(60^\circ) = (5)(10)\frac{\sqrt{3}}{2} = 25\sqrt{3}$. To determine the direction of ***u***×***v*** you can use the Right Hand Rule. Curl the fingers of your right hand from ***u*** to ***v***. Your thumb will point down as you do this indicating that ***u***×***v*** is directed into the page.

12. **(a)** $\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin\left(\frac{\pi}{2}\right) = (3)(2) = 6$.

(b) Note that the vector ***b*** is parallel to the *z*-axis. As ***a***×***b*** is perpendicular to the vector ***b***, the cross product ***a***×***b*** will lie in the *xy*-plane. This means that the *z*-component of ***a***×***b*** is zero.

As ***a***×***b*** is perpendicular to the vector ***a***, which lies in the first quadrant of the *xy*-plane, the cross product ***a***×***b*** will lie in the fourth quadrant of the *xy*-plane. This means that the *x*-component of ***a***×***b*** will be positive and the *y*-component of ***a***×***b*** will be negative.

16. The cross product of two vectors is perpendicular to both of the two vectors. This means that:

$$\langle 1, 1, 1\rangle \times \langle 2, 0, 1\rangle = \langle 1, 1, -2\rangle$$

is perpendicular to both <1, 1, 1> and <2, 0, 1>. To find two unit vectors that are perpendicular to both <1, 1, 1> and <2, 0, 1>, we will divide <1, 1, −2> by its magnitude (which is $\sqrt{6}$) and then multiply by +1 and −1. The two vectors are:

$$\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle \qquad \text{and} \qquad \frac{-1}{\sqrt{6}}\langle 1, 1, -2\rangle.$$

28. **(a)** To find a non-zero vector orthogonal to the plane formed by the three points $P = (2, 0, -3)$, $Q = (3, 1, 0)$ and $R = (5, 2, 2)$ we will first create two vectors that run between these points, and then cross them. The cross product is the vector we are looking for.

Subtracting P from Q gives the vector <1, 1, 3> and subtracting P from R gives the vector <3, 2, 5>. The vector we want is the cross product:

$$<1, 1, 3>\times<3, 2, 5> = <-1, 4, -1>.$$

(b) The area of the triangle formed by P, Q and R is one half of the area of the parallelogram whose sides are given by <1, 1, 3> and <3, 2, 5>. Therefore:

$$\text{Area} = 0.5\cdot|<1, 1, 3>\times<3, 2, 5>| = \tfrac{3}{2}\sqrt{2}.$$

30. To find the volume of the parallelpiped formed by the vectors <1, 1, −1>, <1, −1, 1> and <−1, 1, 1> we will calculate the absolute value of the scalar triple product of these three vectors. This is:

$$\left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right| = |-4| = 4.$$

The volume of the parallepiped is 4 of whatever volume units are employed.

34. The logic of this problem is as follows: We will use the four given points to construct three vectors (that run between the points). We will then use the scalar triple product to calculate the volume of the parallelpiped that is formed by these three vectors. If we get zero for the volume, then the parallelpiped has zero height. If the parallelpiped has zero height, then all of the three vectors (and hence, all of the four points) must all lie together in a single, flat plane.

Creating the three vectors:

Subtracting A from B gives: <2, −4, 4>.

Subtracting A from C gives: <4, −1, −2>.

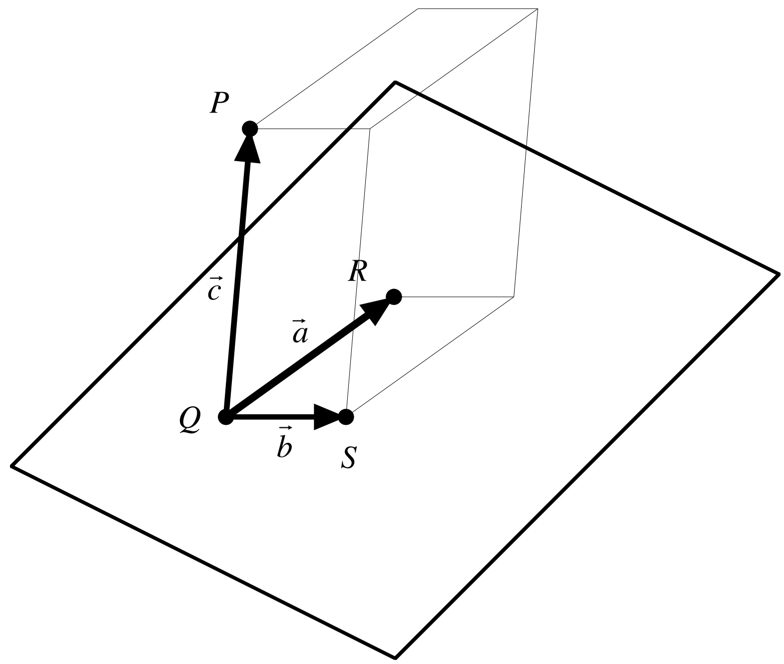
Subtracting A from D gives: <2, 3, −6>.

The scalar triple product of the three vectors is:

$$<2, -4, 4>\bullet(<4, -1, -2>\times<2, 3, -6>) = <2, -4, 4>\bullet<12, -20, 14> = 24 - 80 + 56 = 0.$$

As the scalar triple product is zero, the four points all lie in the same plane.

40. A good place to begin a problem like this is with a sketch to show what is going on. A sketch can help you to visualize what each of the algebraic quantities actually is, and allow you to use your geometric intuition to see why the algebraic quantities are combined in the way that they are in the formula that you have to establish.

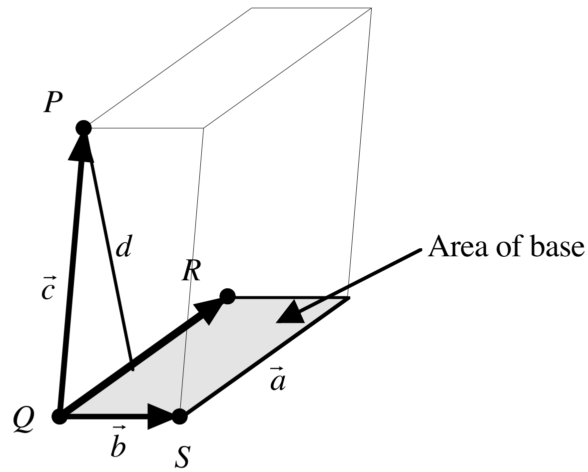


Note the parallelpiped formed by the vectors ***a***, ***b***, and ***c***. Computing the volume of this parallelpiped will be the basis for establishing the formula for the distance from the plane to the point *P*.

On one hand, the volume of the parallelpiped is given by the absolute value of the scalar triple product:

$$\text{Volume} = \left|\left(\vec{a} \times \vec{b}\right) \bullet \vec{c}\right|.$$

On the other hand, the volume of the parallelpiped is equal to the area of the base of the shape multiplied by the perpendicular (i.e. measured perpendicular to the plane) height of the shape, *d*. The perpendicular height of the shape is also the distance between the point *P* and the plane.



The area of the base is the area of the parallelogram whose sides are given by ***a*** and ***b***, and so is equal to |***a***×***b***|. Multiplying this area by d gives that the volume of the parallelepiped is equal to:

$$\text{Volume} = \left|\vec{a} \times \vec{b}\right| \cdot d.$$

Equating the two expressions for the volume of the parallelpiped and solving for the distance d gives:

$$d = \frac{\left|\left(\vec{a} \times \vec{b}\right) \bullet \vec{c}\right|}{\left|\vec{a} \times \vec{b}\right|}.$$

Problems from Pages 551-553 (Section 10.5)

4. To form the equation of the line we want, we need a point on the line (which is given to use as (0, 0, 0)) and a direction vector. As the line we must create is parallel to:

$$\langle x, y, z\rangle = \langle 0, 1, 4\rangle + t \cdot \langle 2, -1, 3\rangle,$$

we can use the vector <2, −1, 3> as our direction vector. The vector equation for our line is therefore:

$$\langle x, y, z\rangle = \langle 0, 0, 0\rangle + t \cdot \langle 2, -1, 3\rangle,$$

and the parametric equations for our line are:

$$x = 2t \qquad y = -t \qquad z = 3t.$$

10. To find a vector equation for the line of intersection of the planes:

$$x + y + z = 1 \qquad \text{and} \qquad x + z = 0$$

we must find a point on the line and the direction vector of the line.

A point on the line is a point (x, y, z) that satisfies both of the planar equations above. Perhaps the simplest point with this property is (0, 1, 0).

The direction of the line is perpendicular to the normal vectors of the two planes. It can be calculated by finding the cross product of the two normal vectors.

$$<1, 1, 1>\times<1, 0, 1> = <1, 0, -1>.$$

The vector equation for the line of intersection is:

$$<x, y, z> = <0, 1, 0> + t\cdot<1, 0, -1>.$$

The parametric equations for the line of intersection are:

$$x = t \qquad y = 1 \qquad z = -t.$$

Rearranging the parametric equations to make t the subject and then equating gives the symmetric equations:

$$x = -z \qquad y = 1.$$

30. To write down an equation for the plane that we want, we need a point that lies in the plane and the normal vector to the plane.

To find a point that lies in the plane, we can find a point on the line of intersection of the two planes:

$$x - z = 1 \qquad \text{and} \qquad y + 2z = 3.$$

There are many such points, one of which is (1, 3, 0).

To find the normal vector to the plane, we will find two vectors that lie in the plane and take their cross product.

As the plane we are creating is perpendicular to the plane $x + y - 2z = 1$, the vector $<1, 1, -2>$ lies in the plane we are creating.

As the line of intersection of the two planes $x - z = 1$ and $y + 2z = 3$ lies in the place we are creating, the direction vector of this line will also lie in the plane we are creating. This direction vector is given by $\langle 1, 0, -1\rangle \times \langle 0, 1, 2\rangle = \langle 1, -2, 1\rangle$.

The normal vector to the plane we are creating will be:

$$\langle 1, 1, -2\rangle \times \langle 1, -2, 1\rangle = \langle 3, 3, 3\rangle.$$

Putting this normal vector together with the point (1, 3, 0) yields the equation for the plane:

$$3\cdot(x - 1) + 3\cdot(y - 3) + 3\cdot(z - 0) = 0.$$