Solutions to Homework #3

Problems from Pages 502-504 (Section 9.3)

18. We will substitute for x and y in the linear equation and then solve for r.

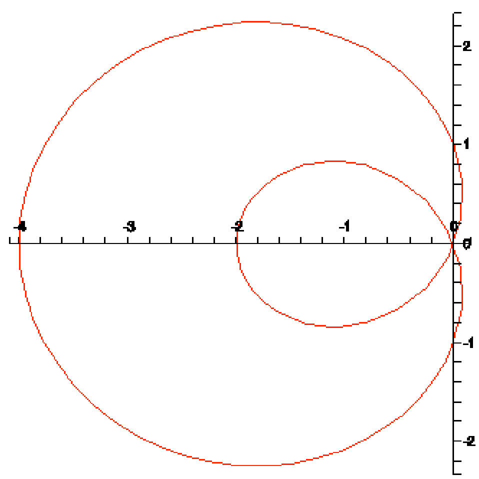
$$x + y = 9$$

$$r \cdot \cos(\theta) + r \cdot \sin(\theta) = 9$$

$$r \cdot (\cos(\theta) + \sin(\theta)) = 9$$

$$r = \frac{9}{\cos(\theta) + \sin(\theta)}.$$

28. A sketch of the curve $r = 1 - 3 \cdot \cos(\theta)$ is shown below.



50. We need to calculate the value of dy/dx starting from the polar equation $r = \sin(3\theta)$ at the point where $\theta = \pi/6$.

To do this, we will set up equations for x and y as functions of θ.

$$x = r \cdot \cos(\theta) = \sin(3\theta) \cdot \cos(\theta).$$

$$y = r \cdot \sin(\theta) = \sin(3\theta) \cdot \sin(\theta).$$

Differentiating these functions with respect to θ gives the following formulas.

$$\frac{dx}{d\theta} = 3 \cdot \cos(3\theta) \cdot \cos(\theta) - \sin(3\theta) \cdot \sin(\theta)$$

$$\frac{dy}{d\theta} = 3 \cdot \cos(3\theta) \cdot \sin(\theta) + \sin(3\theta) \cdot \cos(\theta).$$

Evaluating these derivatives at $\theta = \pi/6$ and finding the quotient gives:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}} = -\sqrt{3}.$$

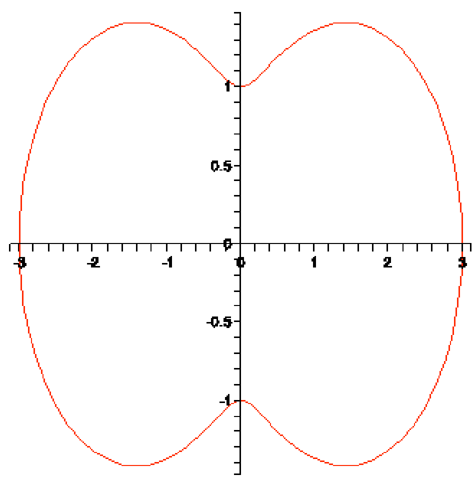
The slope of the tangent line to $r = \sin(3\theta)$ at the point where $\theta = \pi/6$ is $-\sqrt{3}$.

Problems from Pages 508-509 (Section 9.4)

8. The rose has 8 petals, so each petal is $2\pi/8 = \pi/4$ "wide." The limits of integration to give the area of one of the eight petals will be from $\theta = 0$ to $\theta = \pi/4$. The integral giving the area is then:

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2(4\theta)d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 - \cos(8\theta))d\theta = \frac{1}{4}\left[\theta - \frac{1}{8}\sin(8\theta)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{16}.$$

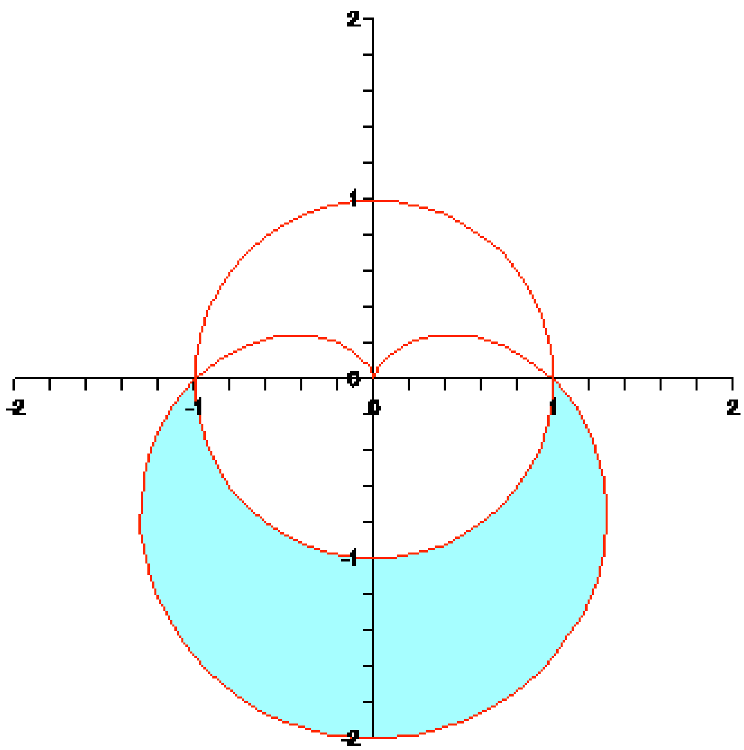
12. A sketch of the curve $r = 2 + \cos(2\cdot\theta)$ (it is shaped a little like a peanut or a butterfly depending on the scales that you use on your axis) is shown below.



The limits of integration for calculating the area of this curve are $\theta = 0$ to $\theta = 2\pi$. Computing the integral that gives the area enclosed by this curve gives:

$$\frac{1}{2}\int_0^{2\pi}\left(2+\cos(2\theta)\right)^2 d\theta = \frac{1}{2}\int_0^{2\pi}\left(4+4\cos(2\theta)+\cos^2(2\theta)\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(4+4\cos(2\theta)+\tfrac{1}{2}+\tfrac{1}{2}\cos(4\theta)\right)d\theta = \frac{9\pi}{2}$$

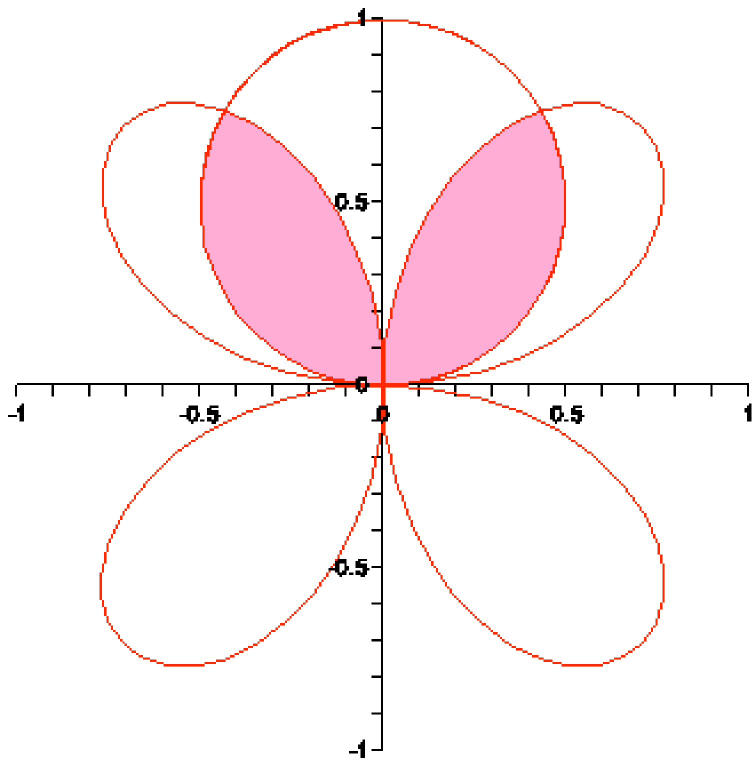
20. The area that is inside $r = 1 - \sin(\theta)$ and $r = 1$ is shaded in the diagram given below. From this diagram we can see that the limits of integration will be $\theta = \pi$ to $\theta = 2\pi$.



The integral that gives this shaded area is:

$$\frac{1}{2}\int_{\pi}^{2\pi}\left(1-\sin(\theta)\right)^2 - 1 d\theta = \frac{1}{2}\int_{\pi}^{2\pi}\left(\sin^2(\theta) - 2\cdot\sin(\theta)\right)d\theta = \frac{\pi}{4} + 2.$$

24. The area consists of two "petals" trapped between $r = \sin(\theta)$ on their underside and $r = \sin(2\theta)$ on their upper side. A sketch showing the two petals is given below.



We will calculate the area of the petal in the positive quadrant and then double this to get the total area.

The curve $r = \sin(\theta)$ leaves the origin at $\theta = 0$ and represents the lower edge of the petal until it intersects the curve $r = \sin(2\theta)$. To determine the value of θ at which the intersection occurs, we will solve the equation:

$$\sin(\theta) = \sin(2\theta).$$

Now, $\sin(2\theta) = 2\cdot\sin(\theta)\cdot\cos(\theta)$ so the above equation can be rearranged to give:

$$\sin(\theta)\cdot[2\cdot\cos(\theta) - 1] = 0.$$

The most important solutions[1] of this equation for this problem are $\theta = \pi/3 + 2n\pi$ (where n is an integer), and the angle that corresponds to the first non-zero intersection of the two polar curves is $\theta = \pi/3$.

After the intersection point, the polar curve $r = \sin(2\theta)$ forms the edge of the petal and remains so until it returns to the origin at $\theta = \pi/2$.

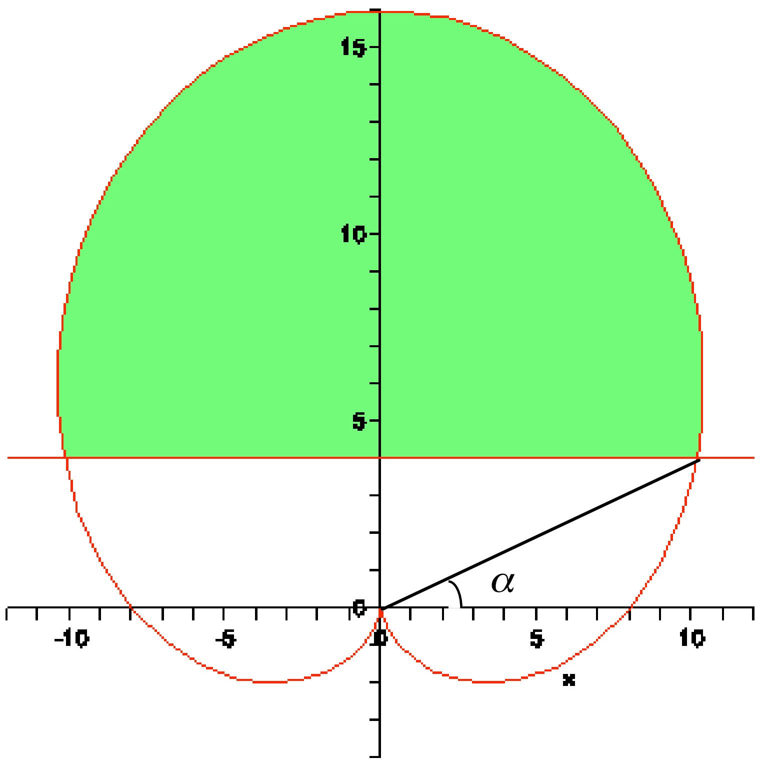
The integrals giving the total area of the petal are:

[1] There are others, such as $\theta = n\cdot\pi$ (where n is an integer), but they are not helpful for solving Problem 24.

$$\frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(\theta)d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2(2\theta)d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos(2\theta))d\theta + \frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos(4\theta))d\theta = \frac{4\pi - 3\sqrt{3}}{32}.$$

The total area that is shaded in the sketch will then be: $\dfrac{4\pi - 3\sqrt{3}}{16}$.

28. The area that we must find is shown in the diagram given below. The upper bound of the area is given by the polar curve $r = 8 + 8\cdot\sin(\theta)$, and the lower edge of the area is given by the horizontal line $y = 4$, which has polar equation $r = \dfrac{4}{\sin(\theta)}$.



The area begins with the angle α shown in the diagram (above) and concludes at the angle $\pi - \alpha$. The integral that will give the area is:

$$\frac{1}{2}\int_{\alpha}^{\pi-\alpha} (8 + 8\cdot\sin(\theta))^2 - \left(\frac{4}{\sin(\theta)}\right)^2 d\theta.$$

To find the value of the angle α we will find the intersection point of the two polar curves. To do this we must solve the equation:

$$8 + 8\cdot\sin(\alpha) = \frac{4}{\sin(\alpha)},$$

which can be rearranged to the quadratic equation:

$$2\cdot\sin^2(\alpha) + 2\cdot\sin(\alpha) - 1 = 0.$$

Using the quadratic formula to solve for $\sin(\alpha)$ gives:

$$\sin(\alpha) = \frac{-1+\sqrt{3}}{2}.$$

(Only one root is listed here as the other has an absolute value greater than one, and so is not relevant to this problem.) Applying the inverse sine function to both sides gives that $\alpha \approx 21.5°$.

Computing the integral for the area with $\alpha \approx 21.5°$ gives an exact answer (which is something of an exercise to obtain) of:

$$48\cdot\pi + 48\cdot\sqrt[4]{12} - 96\cdot\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

Working things out using $\alpha \approx 21.5°$ gives an approximate decimal answer of 204.2 square meters.

Problems from Page 514 (Section 9.5)

10. The equation for this conic can be re-written as follows:

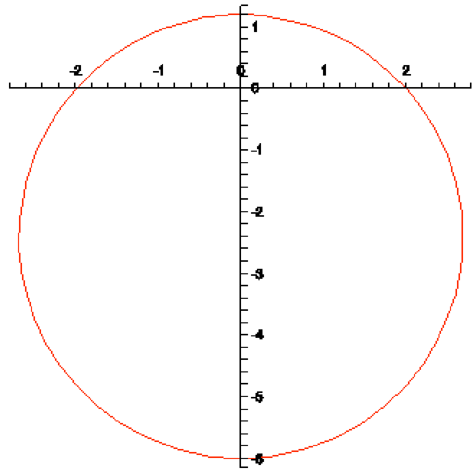
$$r = \frac{6}{3+2\cdot\sin(\theta)} = \frac{2}{1+\frac{2}{3}\cdot\sin(\theta)} = \frac{3\cdot\frac{2}{3}}{1+\frac{2}{3}\cdot\sin(\theta)}.$$

(a) The eccentricity is $e = 2/3$.

(b) The curve is an ellipse.

(c) The equation of the directrix is: $y = 3$.

(d) A sketch of the conic section is given below.



12. The equation for this conic can be re-written as follows:

$$r = \frac{4}{2 - 3 \cdot \cos(\theta)} = \frac{2}{1 - \frac{3}{2} \cdot \cos(\theta)} = \frac{\frac{4}{3} \cdot \frac{3}{2}}{1 - \frac{3}{2} \cdot \cos(\theta)}.$$

(a) The eccentricity is $e = 3/2$.

(b) The curve is a hyperbola.

(c) The equation of the directrix is $y = -4/3$.

(d) A plot showing this curve is shown below. Note that the straight lines in the plot are not part of the conic section, but show the asymptotes that the hyperbola approaches.