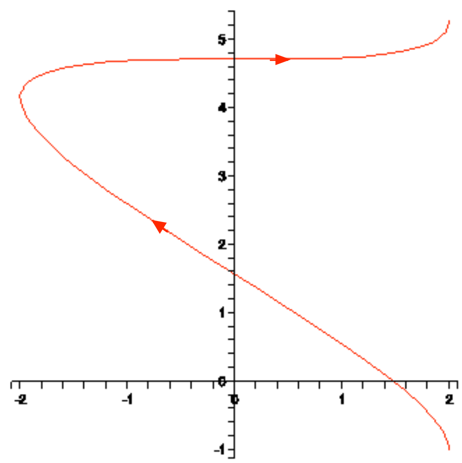
Solutions to Homework #1

Problems from Pages 486-488 (Section 9.1)

2. A table giving some of the values of the curve defined by the parametric equations follows. The curve in the *xy*-plane can be sketched by plotting these points on the *xy*-plane and connecting the points with a smooth curve.

t	0	$\pi/2$	π	$3\pi/2$	2π
x	2	0	-2	0	2
y	-1	$\pi/2$	$\pi + 1$	$3\pi/2$	$2\pi - 1$

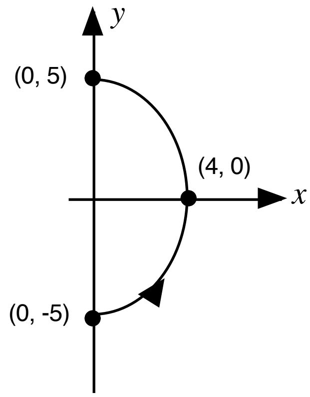


10. **(a)** To obtain a Cartesian equation we can rearrange the given parametric equations to make $\sin(\theta)$ and $\cos(\theta)$ the subjects. We can then square these and add them to get:

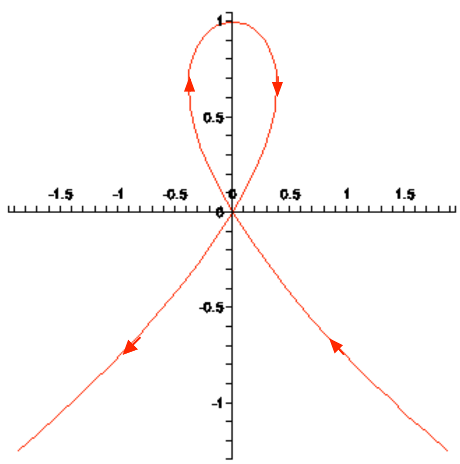
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2(\theta) + \sin^2(\theta) = 1.$$

This is the equation of an ellipse.

(b) When θ ranges from $-\pi/2$ to $\pi/2$, the right hand side of the ellipse will be traced out, starting at the point $(0, -5)$ and rising as shown in the diagram given below.



20. A sketch of the curve is shown below. This curve was obtained by estimating values of x and y from the two graphs that were given on page 486 and then plotting the points on the xy-plane. As soon as enough points had been plotted to clearly suggest the shape of the curve, the points were connected with a smooth curve.



22. **(a)** Matches Graph IV.

(b) Matches Graph VI.

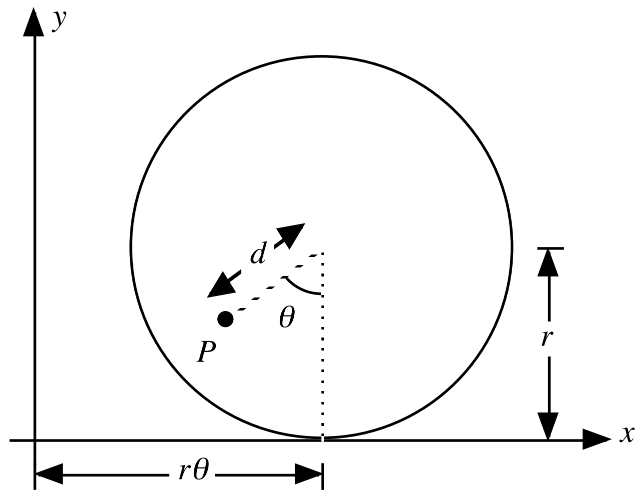
(c) Matches Graph V.

(d) Matches Graph III.

(e) Matches Graph I.

(f) Matches Graph II.

34. The situation after the circle has rolled a horizontal distance of $r\theta$ is shown below.

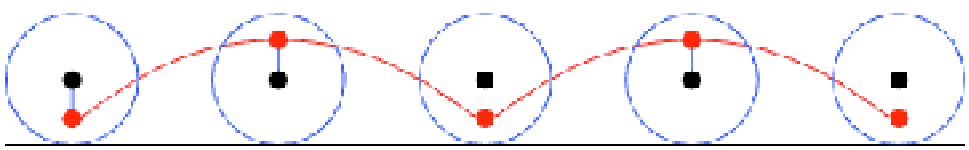


The x and y coordinates of the center of this circle are $x = r\theta$ and $y = r$. Using right angle trigonometry, the horizontal distance between the center of the circle and the point P is given by $d{\cdot}\sin(\theta)$ and the vertical distance between the center of the circle and the point P is given by $d{\cdot}\cos(\theta)$. Subtracting these from the coordinates of the center of the circle gives the parametric equations for the trochoid:

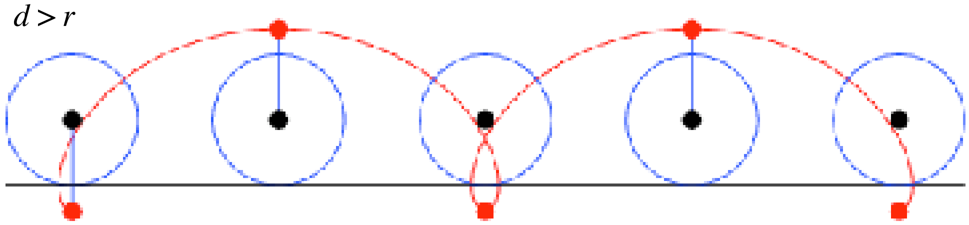
$$x(\theta) = r{\cdot}\theta - d{\cdot}\sin(\theta) \qquad \text{and} \qquad y(\theta) = r - d{\cdot}\cos(\theta).$$

Pictures[1] of trochoids with $d < r$ and $d > r$ are shown below.

$d < r$



$d > r$



[1] Pictures created using an image from: http://mathworld.wolfram.com/

Problems from Pages 494-495 (Section 9.2)

6. To find the equation of the tangent line we will begin by calculating the x and y coordinates of the point of tangency. This is accomplished by plugging $\theta = 0$ into the two parametric equations. Doing this gives $x = 1$ and $y = 1$.

Next we will calculate the derivative $\frac{dy}{dx}$ when $\theta = 0$. To do this we first take the derivatives of $x(\theta)$ and $y(\theta)$.

$$\frac{dx}{d\theta} = -\sin(\theta) + 2\cdot\cos(2\theta) \qquad \text{so that} \qquad \left.\frac{dx}{d\theta}\right|_{\theta=0} = 2, \text{ and,}$$

$$\frac{dy}{d\theta} = \cos(\theta) - 2\cdot\sin(2\theta) \qquad \text{so that} \qquad \left.\frac{dy}{d\theta}\right|_{\theta=0} = 1.$$

Forming the quotient of these gives $\frac{dy}{dx} = \frac{1}{2}$ when $\theta = 0$.

Writing the equation of the tangent line in point-slope form gives the final answer:

$$y - 1 = \tfrac{1}{2}\cdot(x - 1).$$

14. We will begin by calculating the derivatives of the parametric equations $x(t)$ and $y(t)$.

$$\frac{dx}{dt} = 6t^2 + 6t - 12 \qquad \text{and} \qquad \frac{dy}{dt} = 6t^2 + 6t.$$

The values of t for which $\frac{dx}{dt} = 0$ are $t = -2$ and $t = 1$. The values of t for which $\frac{dy}{dt} = 0$ are $t = -1$ and $t = 0$.

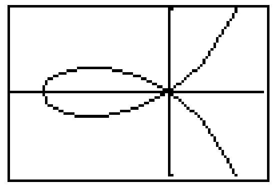
The points on the curves where the tangent line is horizontal are those where $\frac{dy}{dt} = 0$. We can find the coordinates of these points by plugging $t = -1$ and $t = 0$ into the parametric equations $x(t)$ and $y(t)$. Doing this gives:

t	−1	0
x	13	0
y	2	1

The points on the curves where the tangent line is vertical are those where $\frac{dx}{dt} = 0$. We can find the coordinates of these points by plugging $t = -2$ and $t = 1$ into the parametric equations $x(t)$ and $y(t)$. Doing this gives:

t	−2	1
x	28	−7
y	−3	6

22. If we graph the curve defined by the two parametric equations on a calculator, we get a graph that resembles the one shown below.



From this graph, we can see that the point where the graph crosses itself is the point where $x = 0$ and $y = 0$. To determine the tangent lines to this point, we need to know the values of t that correspond to $x = 0$ and $y = 0$. We can find these values of t by solving the equation:

$$x(t) = 1 - 2 \cdot \cos^2(t) = 0.$$

These are $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$. Next we will calculate the derivative dy/dx using each of these two values of t. To do this we must first calculate the derivatives:

$$\frac{dx}{dt} = 2 \cdot \sin(t) \cdot \cos(t) \quad \text{and} \quad \frac{dy}{dt} = 2 \cdot \sin(2t) \cdot \tan(t) - \cos(2t) \cdot \sec^2(t)$$

Substituting the two values of t into these derivatives and forming the quotient of these gives:

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = 1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{4}} = -1.$$

The two tangent lines that meet the parametric curve at the point of self-intersection are therefore $y = x$ and $y = -x$.

24. **(a)** To find a formula for dy/dx we need to calculate $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$. Doing this gives:

$$\frac{dx}{d\theta} = -3a \cdot \cos^2(\theta) \cdot \sin(\theta) \quad \text{and} \quad \frac{dy}{d\theta} = 3a \cdot \sin^2(\theta) \cdot \cos(\theta).$$

Forming the quotient of these and simplifying gives the formula for the derivative:

$$\frac{dy}{dx} = \frac{3a \cdot \sin^2(\theta) \cdot \cos(\theta)}{-3a \cdot \cos^2(\theta) \cdot \sin(\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta).$$

(b) The tangent line will be horizontal when $\tan(\theta) = 0$. The solutions of this equation are $\theta = n\pi$, where n is an integer. Plugging these values of θ into the parametric equations $x(t)$ and $y(t)$ gives:

$$x(n\pi) = \begin{cases} a & ,n \text{ is even} \\ -a & ,n \text{ is odd} \end{cases}$$

and $y(n\pi) = 0$. The points at which the tangent lines are horizontal are $(x, y) = (\pm a, 0)$.

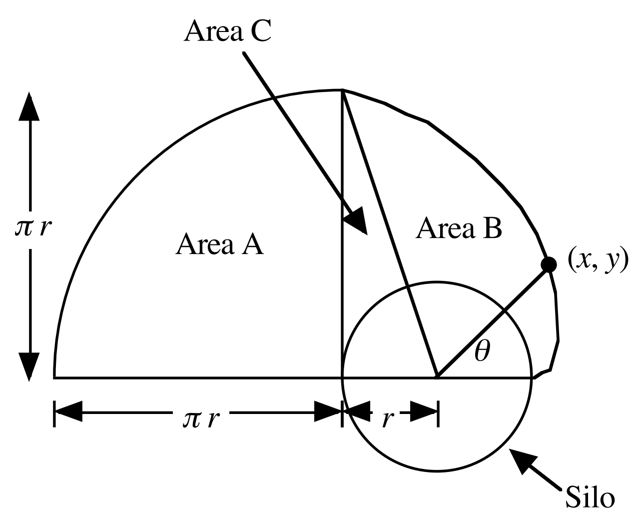
The tangent line will be vertical when $\cos(\theta) = 0$. The solutions of this equation are $\theta = n\pi/2$ where n is an odd integer. Plugging this value of θ into the parametric equations $x(\theta)$ and $y(\theta)$ gives $x = 0$ and $y = \pm a$ so that the points on the curve where the tangent lines are horizontal are $(x, y) = (0, \pm a)$.

(c) The points at which the tangent line has slope equal to $+1$ or -1 are the points given by the values of θ for which $\tan(\theta) = 1$ or $\tan(\theta) = -1$. The values of θ that satisfy these equations are $\theta = n\pi/4$ where n is an odd integer. Plugging these values into the parametric equations for $x(\theta)$ and $y(\theta)$ gives:

$$x(n\pi/4) = \pm a\cdot(1/2)^{3/2} \qquad \text{and} \qquad y(n\pi/4) = \pm a\cdot(1/2)^{3/2}$$

with the + or – depending on the specific value of n.

54. A sketch showing half of the area available to the cow is shown in the diagram below. This area has been broken into three pieces. Note that the sum of these three areas also includes the area of half the silo (which is $\pi r^2/2$) and this will need to be subtracted to get the grazing area available to the cow.



Area A is one quarter of a circle with radius πr. The area of this quarter circle will be $\pi^3 r^2/4$.

The rest of the curve defining the area that the cow can graze is given by a pair of parametric equations with parameter θ. These equations are the ones derived in Problem 53 on page 495 of the textbook, namely:

$$x = r\cdot(\cos(\theta) + \theta\cdot\sin(\theta)) \qquad \text{and} \qquad y = r\cdot(\sin(\theta) - \theta\cdot\cos(\theta))$$

for $0 \le \theta \le \tan^{-1}(\pi)$. Using these parametric equations, Area B is given by the integral:

$$\text{Area B} = \frac{1}{2}\int_0^{\pi-\tan^{-1}(\pi)} r^2\cdot\left(\cos(\theta)+\theta\cdot\sin(\theta)\right)^2 + r^2\cdot\left(\sin(\theta)-\theta\cdot\cos(\theta)\right)^2\cdot d\theta = \frac{\pi^3\cdot r^2}{6}.$$

Area C is the area of a triangle with base r and height πr, so Area C is equal to $\pi r^2/2$.

Adding these three areas together and subtracting half of the area of the silo gives:

$$\text{Half of grazing area} = \pi^3 r^2/4 + \pi^3 r^2/6 + \pi r^2/2 - \pi r^2/2 = 5\pi^3 r^2/12.$$

So the total grazing area is twice this, namely $5\pi^3 r^2/6$.