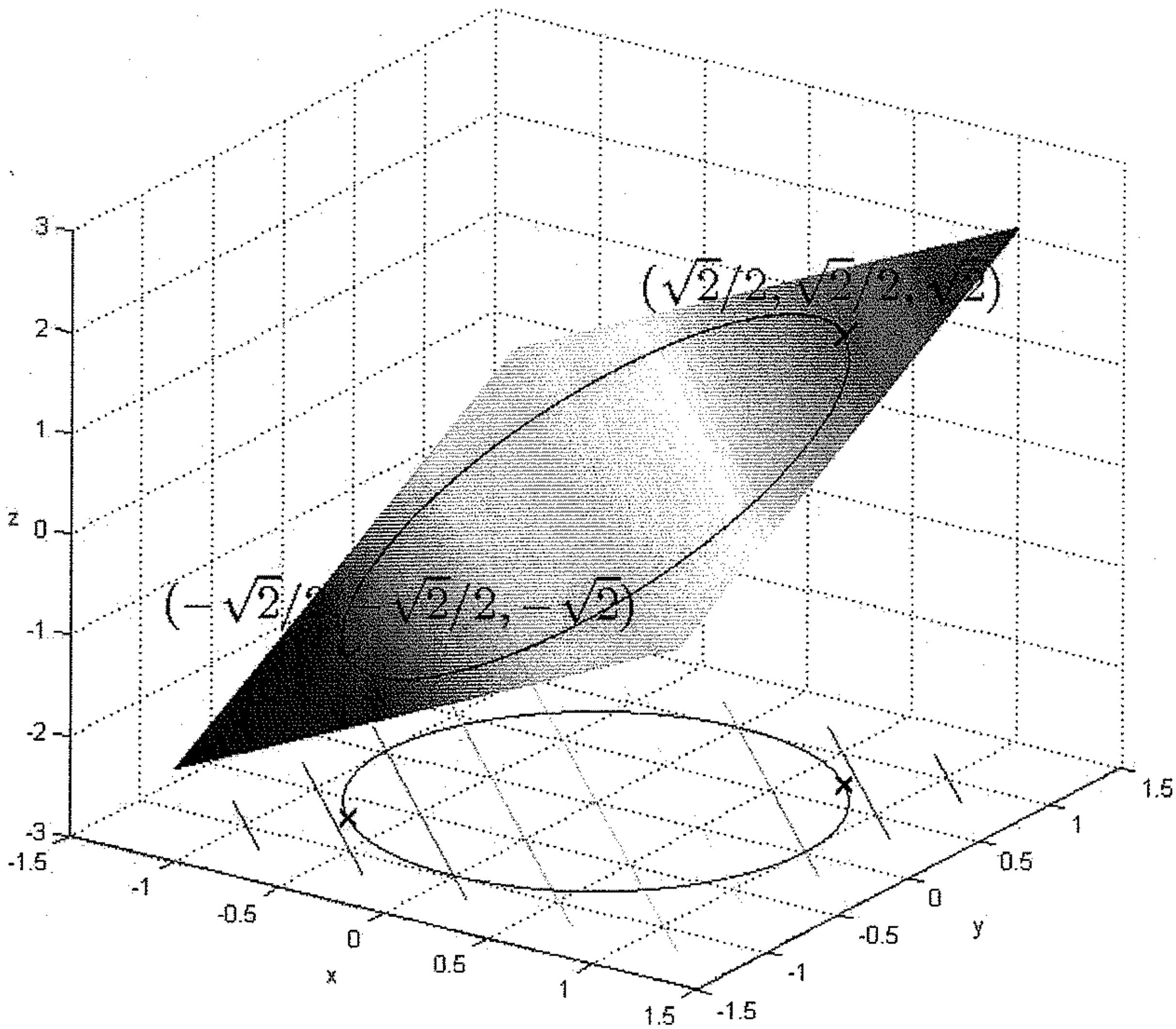
SOLUTIONS

Recitation Handout 9: Lagrange Multipliers

The method of Lagrange Multipliers is an excellent technique for finding the global maximum and global minimum values of a function $f(x, y)$ when the values of x and y that need to be considered are subject to some form of constraint, usually expressed as an equation $g(x, y) = 0$.

The image included here, for example, shows a graphical solution of the constrained optimization problem: Find the global maximum and global minimum of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 1$ (or $g(x, y) = x^2 + y^2 - 1 = 0$).



The constraint $x^2 + y^2 = 1$ can be visualized as a circle in the xy-plane which is then projected onto the graph of $z = f(x, y) = x + y$. The points we are looking for are the points on this projected circle that have the highest and lowest z-values.

As indicated in the diagram, these are the global maximum $z = \sqrt{2}$ (which is attained at the point $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$) and the global minimum $z = -\sqrt{2}$ (which is attained at the point $(x, y) = (-\sqrt{2}/2, -\sqrt{2}/2)$).

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Steps in Solving a Problem Using Lagrange Multipliers

To solve a Lagrange Multiplier problem to find the global maximum and global minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$, you can find the following steps.

Step 1: Calculate the gradient vectors ∇f and ∇g.

Step 2: Write out the system of equations $\nabla f = \lambda \cdot \nabla g$.

Step 3: Solve the system of equations to find **all** points (x, y) that satisfy $\nabla f = \lambda \cdot \nabla g$ **and** $g(x, y) = 0$.

Step 4: Evaluate $f(x, y)$ at each of the points found in Step 3. The largest value of f obtained will be the global maximum and the smallest value of f obtained will be the global minimum.

Example

Find the global maximum and global minimum of $f(x, y) = x^2 \cdot y$ subject to the constraint:

$$x^2 + y^2 = 3.$$

Solution

In this problem, $g(x, y) = x^2 + y^2 - 3$, so that:

$$\nabla f = \langle 2xy, x^2 \rangle \qquad \text{and} \qquad \nabla g = \langle 2x, 2y \rangle.$$

The system of equations $\nabla f = \lambda \cdot \nabla g$ is equivalent to:

$$2xy = 2\lambda x \qquad \text{and} \qquad x^2 = 2\lambda y.$$

If $\lambda = 0$ then $x^2 = 0$ so that $x = 0$. Substituting this into the constraint equation given that $y = \pm\sqrt{3}$. Next, if $\lambda \neq 0$ then $x^2 = 2y^2$. Substituting this into the constraint equation gives $y = \pm 1$ and $x = \pm\sqrt{2}$. The collection of points that solve the Lagrange multiplier equations and the values of the function f are summarized in the table given below.

x	y	f(x, y)
$\sqrt{2}$	1	2
$-\sqrt{2}$	1	2
$\sqrt{2}$	-1	-2
$-\sqrt{2}$	-1	-2
0	$\sqrt{3}$	0
0	$-\sqrt{3}$	0

The global maximum of f is 2 and the global minimum of f is 0. This problem and these points are illustrated on the next page.

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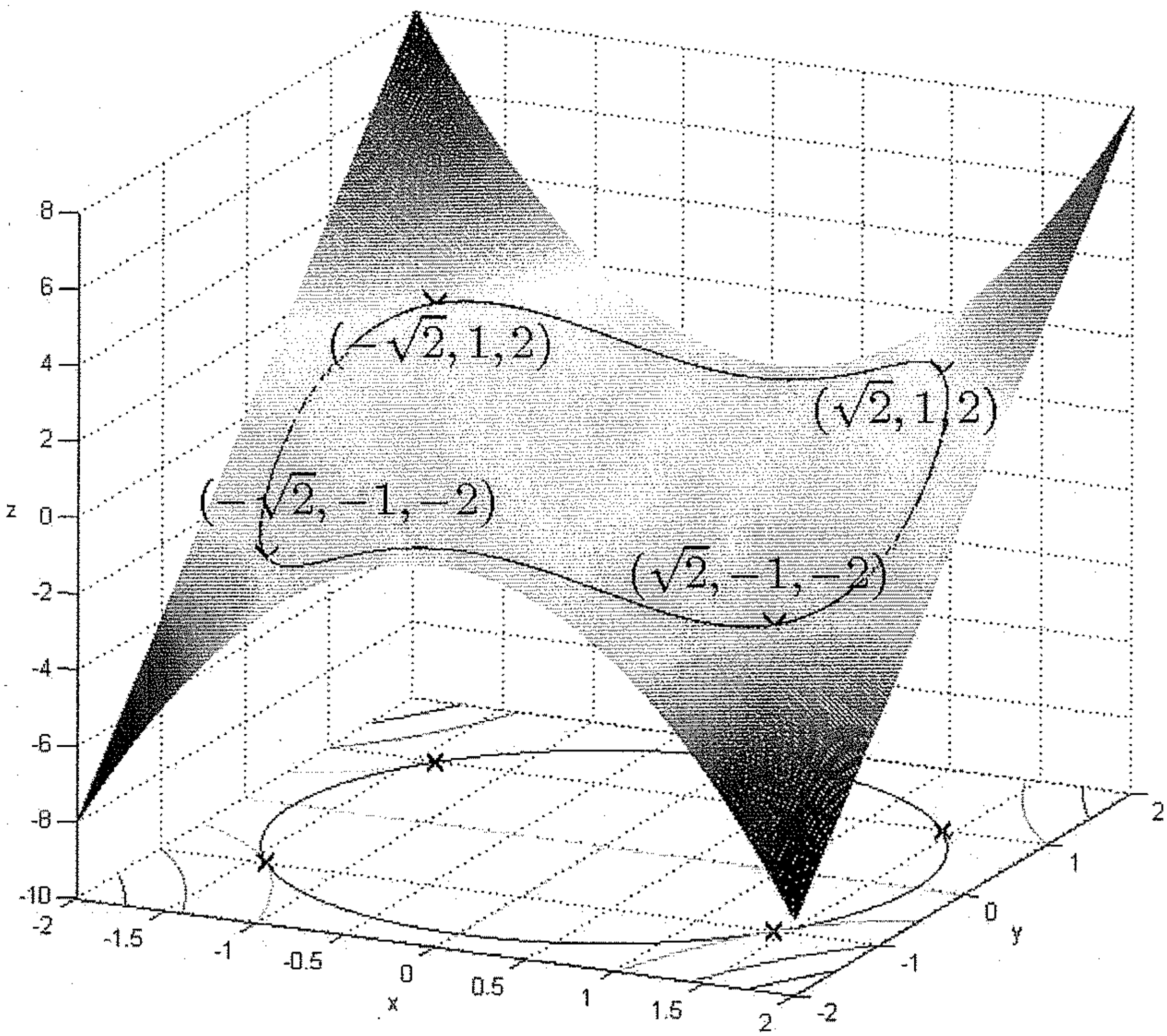

Note that (as indicated by the table) the global maximum and global minimum are not attained at unique points. In Lagrange multiplier problems with a certain level of geometrical symmetry, it is common for the global maximum and global minimum of f to be attained at several points that satisfy $g(x, y) = 0$.

Example

In this example you will calculate the minimum distance from the rectangular hyperbola

$$xy = 1$$

to the origin. (Because this problem probably appeals very strongly to your geometric intuition, you will have a good way to check if your final answer is correct.) Here is the problem expressed as a Lagrange multipliers problem:

Find: The global minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = xy - 1 = 0$.

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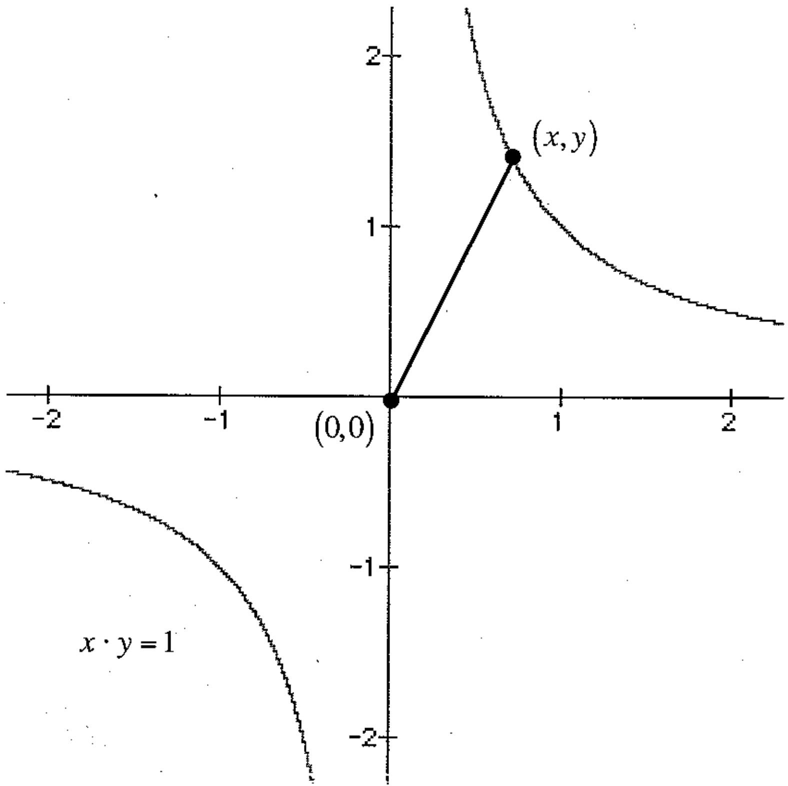
Solution

1. Use the diagram provided below to explain why finding the global minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = xy - 1 = 0$ will calculate the minimum distance between the hyperbola and the origin (0, 0).



The distance between (x,y) and $(0,0)$ is:

$$d = \sqrt{x^2 + y^2}.$$

The global minimum of this function will be attained at the same point(s) where the global minimum of:

$$f(x, y) = x^2 + y^2 = d^2.$$

2. Calculate the two gradient vectors:

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla g = \langle y, x \rangle$$

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3. Create and solve the system of equations $\nabla f = \lambda \cdot \nabla g$.

$$2x = \lambda y \quad \ldots \text{①}$$

$$2y = \lambda x \quad \ldots \text{②}$$

If we multiply ① by x and ② by y we get:

$$2x^2 = \lambda x y = 2y^2$$

or $x^2 = y^2$. Since $x \cdot y = 1$, both x and y must have the same sign so $x = y$.

Substituting $x = y$ into $x \cdot y = 1$ gives $x^2 = 1$ so $x = \pm 1$ and $y = \pm 1$.

4. Copy the solutions of $\nabla f = \lambda \cdot \nabla g$ that you found in Question 3 into the table given below and evaluate $f(x, y) = x^2 + y^2$ at each point.

x	y	$f(x, y)$
1	1	2
−1	−1	2

5. What is the minimum distance from the hyperbola to the point (0, 0)? Does this agree with your geometrical intuition?

The minimum distance is $\sqrt{2}$.

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Lagrange Multipliers in Three Dimensions

When solving a Lagrange multiplier problem for a function with three input values, say $f(x, y, z)$, and a constraint equation that also involves the three variables, say $g(x, y, z) = 0$, the steps that you carry out are exactly the same. However, now you will have one additional equation to solve, namely:

$$\frac{\partial f}{\partial z} = \lambda \cdot \frac{\partial g}{\partial z}.$$

Example

In this example you will solve a problem of classical geometry – what is the volume of the largest rectangular box that can fit entirely within a given closed surface. In this case, the closed surface will be the ellipsoid:

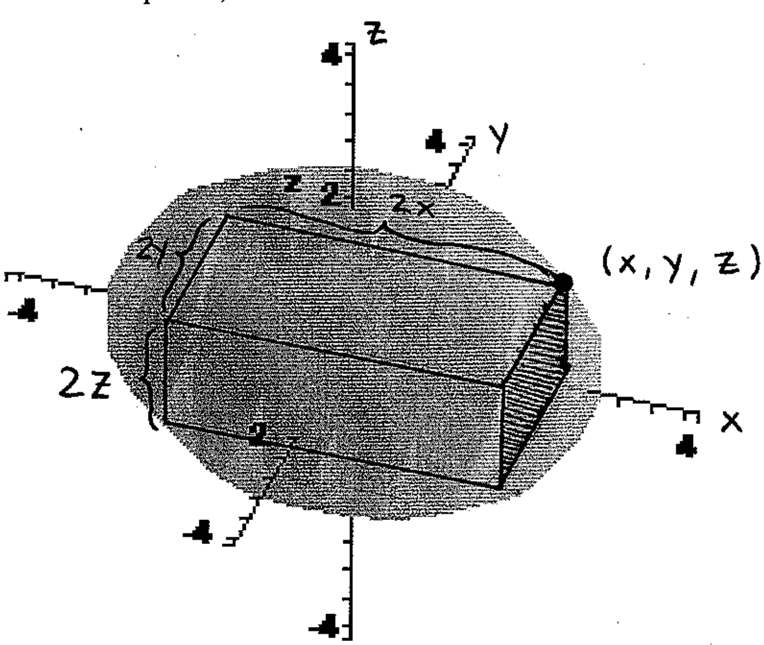
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution

6. The object of this calculation will be to find the global maximum of the function:

$$f(x,y,z) = 8xyz.$$

Use the diagram given below to explain why this formula will give the volume of a rectangular box that just fits inside the ellipsoid. (**Hint:** Use (x, y, z) to represent the coordinates of the point in the positive octant where the corner of the box touches the ellipsoid.)



If the corner of the box is located at (x, y, z) and the box is symmetric about the axes, then the lengths of the sides of the box will be $2x$, $2y$, $2z$.

The volume of the box is then: $V = (2x)(2y)(2z)$
$= 8xyz.$

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7. What is the constraint function $g(x, y, z)$ in this problem?

$$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

8. Calculate the two gradient vectors:

$$\nabla f = \langle 8yz, \; 8xz, \; 8xy \rangle$$

$$\nabla g = \left\langle \frac{2x}{a^2}, \; \frac{2y}{b^2}, \; \frac{2z}{c^2} \right\rangle$$

9. Create and solve the system of equations $\nabla f = \lambda \cdot \nabla g$. (Additional space is provided on the next page.)

$$8yz = \frac{2\lambda}{a^2} x \quad \cdots \quad ①$$

$$8xz = \frac{2\lambda}{b^2} y \quad \cdots \quad ②$$

$$8xy = \frac{2\lambda}{c^2} z \quad \cdots \quad ③$$

Multiply ① by x, ② by y and ③ by z to get:

$$2\lambda \frac{x^2}{a^2} = 2\lambda \frac{y^2}{b^2} = 2\lambda \frac{z^2}{c^2} = 8xyz$$

Now, $\lambda = 0$ will give $x = y = z = 0$ which does not correspond to maximum volume. So, we will assume $\lambda \neq 0$. In this case:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

Substituting these into $g(x, y, z) = 0$ gives:

$$\frac{x^2}{a^2} = \frac{1}{3} \qquad \frac{y^2}{b^2} = \frac{1}{3} \qquad \frac{z^2}{c^2} = \frac{1}{3}.$$

If $x > 0$, $y > 0$, $z > 0$ then:

$$x = \frac{a}{\sqrt{3}} \qquad y = \frac{b}{\sqrt{3}} \qquad z = \frac{c}{\sqrt{3}}$$

and maximum volume is: $V = \dfrac{8}{3\sqrt{3}} abc.$

10. Copy the solutions of $\nabla f = \lambda \cdot \nabla g$ that you found in Question 3 into the table given below and evaluate $f(x, y, z)$ at each point.

x	y	z	$f(x, y, z)$
$a/\sqrt{3}$	$b/\sqrt{3}$	$c/\sqrt{3}$	$\frac{8}{3\sqrt{3}} \cdot abc$