SOLUTIONS

Recitation Handout 8: Finding Global Maximum and Minimum Values

In lecture and in homework we have mainly concentrated on the problem of locating maximum and minimum values by finding the points at which the tangent plane to a surface is horizontal.

This is certainly an important skill. However, in many "real world" problems, it may not be realistic to allow x and y to have any value. For example, if x and y represent quantities of a resource then it might be very reasonable to require that $x \geq 0$ and $y \geq 0$. If the x and y coordinates of a local maximum turn out to be negative, then this will be of no help in obtaining a realistic solution to the problem.

Finding the global maximum and global minimum of a function is all about finding the largest (and smallest) values of the function that are possible when constraints are put upon the x and y values that can be considered. In today's recitation you will have the opportunity to learn:

- What the terms "global maximum" and "global minimum" mean in the context of functions with several variables.
- The kinds of points where the global maximum and global minimum of a function $z = f(x, y)$ may occur.
- A plan of attack for locating the global maximum and global minimum of a function for a given region R of the xy-plane.

Finding the Critical Points of a Function of Several Variables

The purpose of this first part of the recitation is to practice finding the critical points where both partial derivatives are equal to zero, and then to use a graph of the function to classify the critical points as local maximums, local minimums or saddle points.

1. Find the x, y and z coordinates of all critical points of the function:

$$z = f(x,y) = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4 - x^2.$$

Record your results in the table provided on the next page.

$$\frac{\partial f}{\partial x} = -2x$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}y + \frac{1}{8}y^2 - \frac{1}{8}y^3 = -\frac{1}{8}y(y+3)(y-4)$$

The three points where the partial derivatives are both zero are: $(0,0)$, $(0,-3)$ and $(0,4)$.

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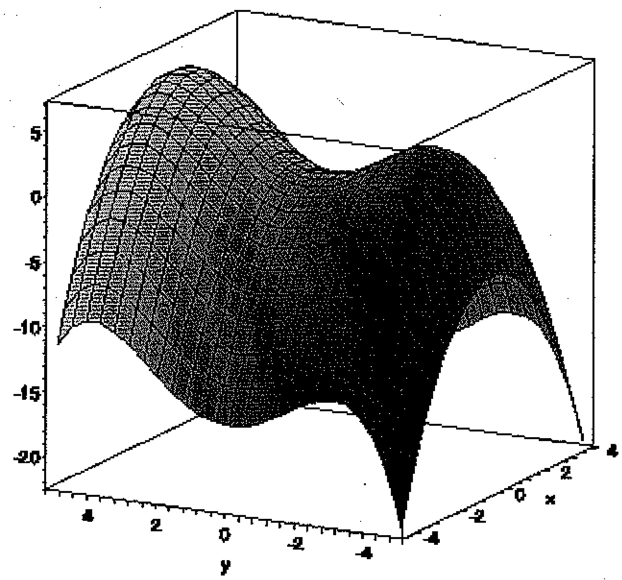
$$z = f(x,y) = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4 - x^2.$$

x	y	z
0	-3	99/32
0	0	0
0	4	20/3

2. The graph shown below is the graph of the function $z = f(x,y) = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4 - x^2$. Use this graph to classify each of the critical points that you found in Question 1 and record your results in the table (below).



x	y	z	Classification
0	-3	99/32	Local maximum
0	0	0	Saddle point
0	4	20/3	Local maximum

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Locations of Global Maximums and Global Minimums

Suppose that R is a set of points (or **region**) of the xy-plane and that $f(x, y)$ is a continuous function whose domain includes R.

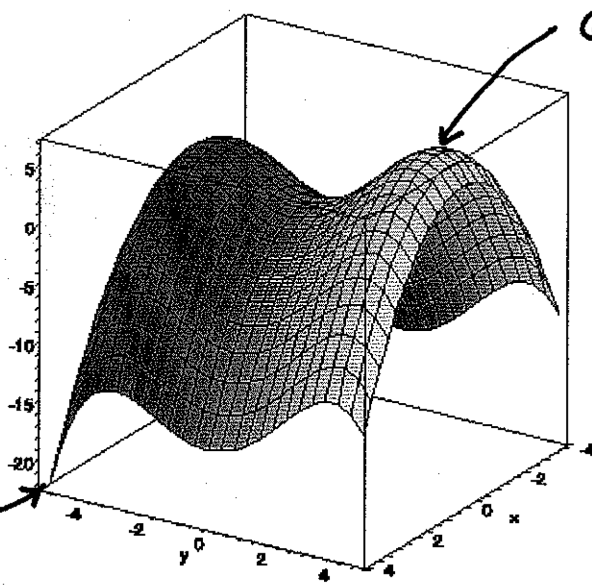
The **global maximum** value of the function $f(x, y)$ attained on the region R is the largest possible value of $z = f(x, y)$ that can be calculated by plugging a point (x, y) from the region R into the formula for $f(x, y)$.

The **global maximum** value of the function $f(x, y)$ attained on the region R is the lowest[1] possible value of $z = f(x, y)$ that can be calculated by plugging a point (x, y) from the region R into the formula for $f(x, y)$.

3. The graph given below shows all points on the surface $z = f(x, y)$ over the rectangular region of the xy-plane that consists of:

$$-4 \le x \le 4 \qquad \text{and} \qquad -5 \le y \le 5.$$

Global maximum
$z = 20/3$



Global minimum
$x = \pm 4 \quad y = -5$
$z \approx -21.9895833$

(This is the same function and the same set of x and y values as the previous graph. This graph is shown from a different point of view so that you can see the side of the graph that was hidden previously.)

Use the table that you created in Question 2 and the two graphs to find the **global maximum value** of $f(x, y)$ over the rectangular region:

$$-4 \le x \le 4 \qquad \text{and} \qquad -5 \le y \le 5.$$

and the **global minimum value** of $f(x, y)$ over the same rectangular region.

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[1] Or most negative is you are not used to thinking of -100 as lower than -1, for instance.

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4. Make a conjecture (educated guess) about the points where the global maximum or global minimum value of a function occurs. That is, at which points of a region R in the xy-plane could the global maximum and global minimum of a continuous function $f(x, y)$ be attained?

It appears that the global maximum and minimum occur either at points where both partial derivatives are equal to zero or points on the boundary (edge) of the region.

Refining the Conjecture

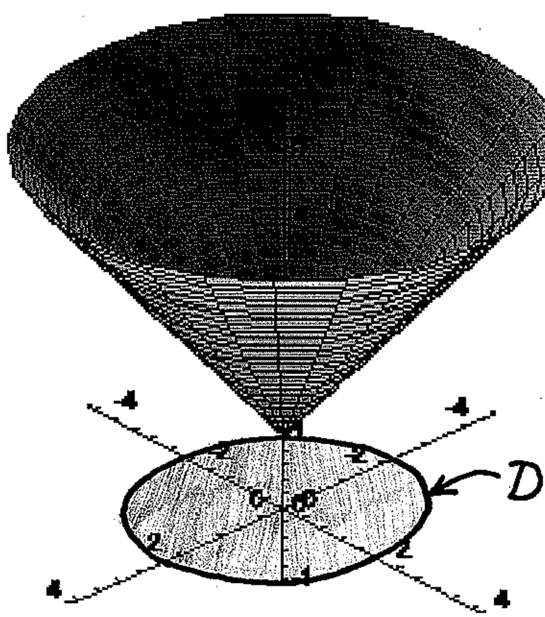
5. The graph of the function

$$z = g(x,y) = 1 + \sqrt{x^2 + y^2},$$

is shown below. Suppose we only consider the region D of the xy-plane for which

$$\sqrt{x^2 + y^2} \le 2.$$

Complete the graph shown below by shading in the points of the xy-plane that make up the region D.



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6. Using the graph shown on the previous page and the formula for $g(x, y)$, find the global maximum and global minimum values of $g(x, y)$ on the region D.

Global minimum occurs at $(0, 0)$.

$z = g(0, 0) = 1.$

Global maximum occurs at any point (x, y) satisfying: $\sqrt{x^2 + y^2} = 2.$

$z = g(x, y) = 3.$

7. Are the locations of both global maximum and global minimum consistent with the prediction of your conjecture from Question 4? If not, adapt or refine your conjecture to take this new example into account.

No, the location of the global minimum is not.

The global max and/or global min may occur at a point where one (or both) of the partial derivatives are undefined.

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Finding the Global Maximum or Minimum of a Function

It is possible to prove a very general mathematical theorem[2] that helps to locate the global maximum and global minimum of a continuous[3] function $f(x, y)$ on a region of the xx-plane, R, that is enclosed by a closed curve C. (A *closed* curve is one that begins and ends at the same point, like a circle.) This theorem lays out all of the points in the region R at which the global maximum and global minimum of $f(x, y)$ can occur.

THEOREM: Suppose that $f(x, y)$ is continuous on the region R of the xy-plane that consists of the points on and within the closed curve C. If $f(a, b)$ is either the global maximum or the global minimum of $f(x, y)$ on R, then one of the following must be true:

(I) (a, b) is a point within C at which both partial derivatives f_x and f_y are zero, or,

(II) (a, b) is a point within C at which at least one of the partial derivatives f_x or f_y is undefined, or,

(III) (a, b) lies on the closed curve C.

The reason that this theorem can be helpful to us is because it gives us a concrete plan of attack for finding the global maximum and global minimum of a function. This plan of attack could be summed up as follows.

STEP 1: Locate the points that lie in the region R where $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

STEP 2: Locate the points that lie in the region R where at least one partial derivative $f_x(x, y)$ or $f_y(x, y)$ is undefined.

STEP 3: Find the points on the boundary of R (that is, the curve C) where maximum or minimum values of $f(x, y)$ are attained.

STEP 4: Evaluate $z = f(x, y)$ at each of the points you have found. The largest z value you get is the global maximum of $f(x, y)$ on R and the smallest z value that you get is the global maximum of $f(x, y)$ on R.

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[2] To state and understand this theorem fully requires knowledge of a field of mathematics called *topology*, and in particular, the topological notion of a compact set. This is beyond the scope of this course, but well worth further investigation if you are interested in the purer side of mathematics. To figure out what is going on you will need to learn: (a) what open and closed sets are, (b) what open covers and finite subcovers of a set are, and (c) the topological definition of continuity. Putting all of these concepts together gives a simple (to state, not so simple to prove) but powerful result: A continuous function always attains a global maximum and a global minimum on a compact set.

[3] In this context, continuous means that the limit of $f(x, y)$ as $(x, y) \to (x_0, y_0)$ is $f(x_0, y_0)$.

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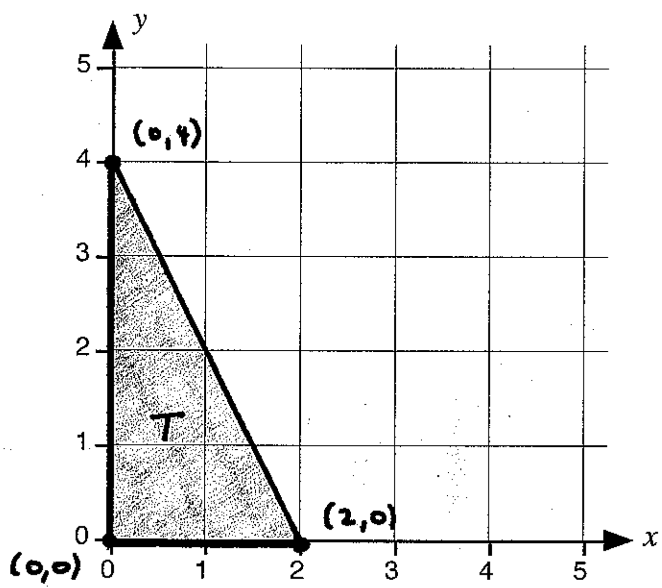
An Example

In this example, you will find the global maximum and minimum values attained by the function:

$$p(x, y) = xy - x - y + 3$$

at points of the triangular region T of the xy-plane with vertices at (0, 0), (2, 0) and (0, 4).

8. Use the axes provided below to sketch the region T in the xy-plane.



9. Find the x and y coordinates of all points in T for which both $p_x(x, y) = 0$ and $p_y(x, y) = 0$. (No need to use the Jacobian determinant and p_{xx} to classify them.) Record your results in the table given below.

$$\frac{\partial p}{\partial x} = y - 1$$

$$\frac{\partial p}{\partial y} = x - 1$$

The only point where both partial derivatives are equal to zero is $(x,y) = (1,1)$.

x	y	z
1	1	2

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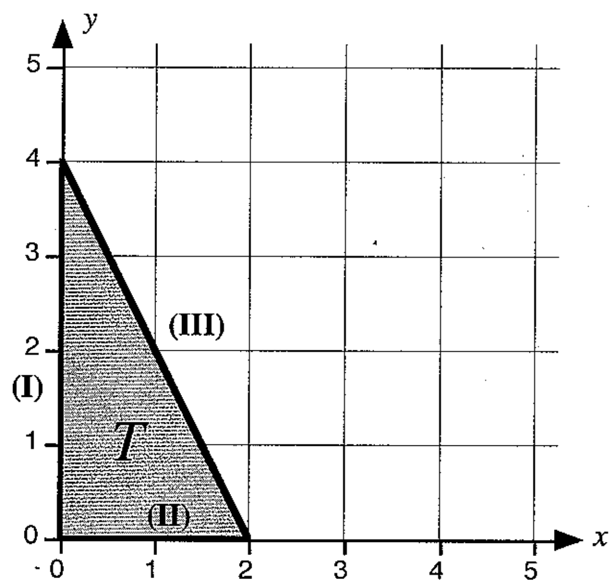
10. Find the x and y coordinates of all points in T for which either $p_x(x, y)$ or $p_y(x, y)$ is undefined. Record your results in the table given below.

The partial derivatives are never undefined.

x	y	z

The curve C that makes up the boundary of the region T can be broken down into three linear segments. These are:



(I) $x = 0$

(II) $y = 0$

(III) $y = -2x + 4$

11. Plug the equation for Curve (I) into the formula for

$$p(x, y) = xy - x - y + 3$$

to give a function of one variable (say $p_1(y)$) only. Find the global maximum and global minimum of $p_1(y)$ on the interval $0 \le y \le 4$. Look at your sketch of the region T. Why is the interval $0 \le y \le 4$ used here?

$P_I(y) = -y + 3.$ Global max : 3

Global min : $-1.$

This interval is the set of y-values that comprise side (I) of the region T.

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12. Plug the equation for Curve (II) into the formula for

$$p(x, y) = xy - x - y + 3$$

to give a function of one variable (say $p_{II}(x)$) only. Find the global maximum and global minimum of $p_{II}(x)$ on the interval $0 \le x \le 2$. Look at your sketch of the region T. Why is the interval $0 \le x \le 2$ used here?

$P_{II}(x) = -x + 3$ Globol max: 3 Global min: 1.

The interval $0 \le x \le 2$ corresponds to the x-values on Side (II) of the region T.

13. Plug the equation for Curve (III) into the formula for

$$p(x, y) = xy - x - y + 3$$

to give a function of one variable (say $p_{III}(x)$) only. Find the global maximum and global minimum of $p_{III}(x)$ on the interval $0 \le x \le 2$. Look at your sketch of the region T. Why is the interval $0 \le x \le 2$ used here?

$$P_{III}(x) = x(-2x+4) - x - (-2x+4) + 3$$
$$= -2x^2 + 5x - 1.$$

$P_{III}'(x) = -4x + 5 = 0$ when $x = 5/4$.

$P_{III}(0) = -1$ $P_{III}(2) = 1$ $P_{III}(5/4) = 2.125$

Global max : 2.125

Global min : -1

14. Using the results that you have found in Questions 9, 10, 11, 12 and 13, find the global maximum and global minimum attained by $p(x, y)$ on the region T.

Global max of $p(x,y)$ on T is 3.

Global min of $p(x,y)$ on T is -1.