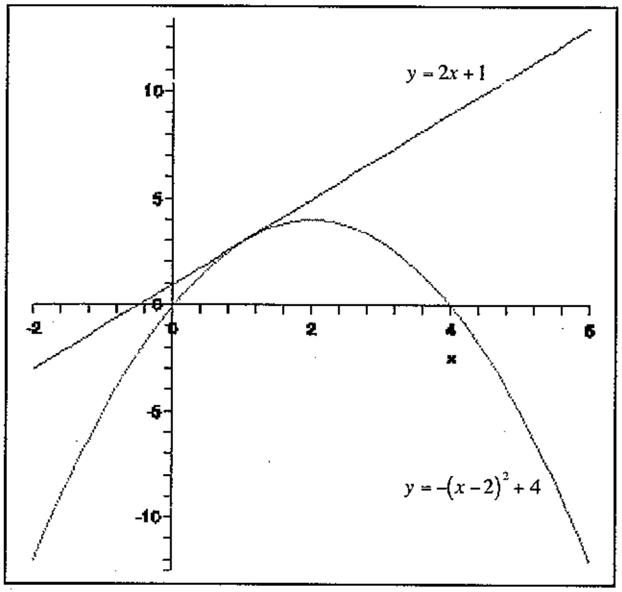
SOLUTIONS

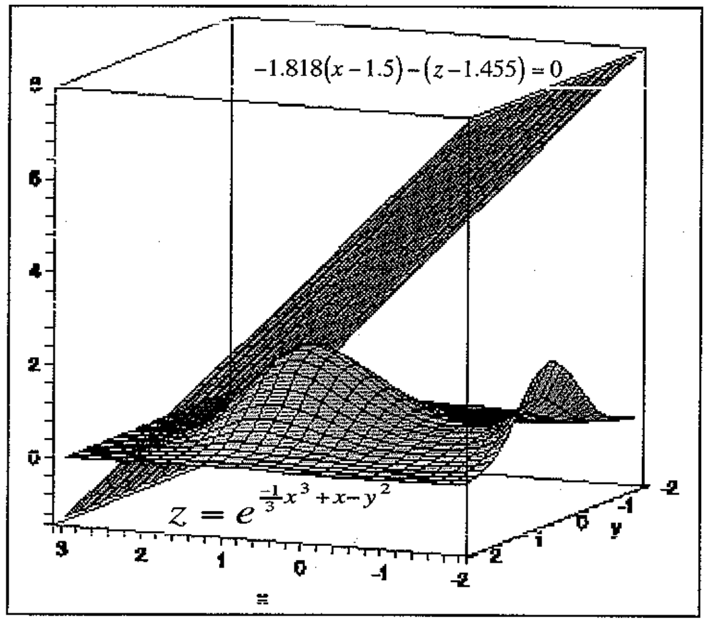
Math 259 **Winter 2009**

Recitation Handout 7: Visualizing a Linear Approximation



In calculus we have very good tools to visualize the tangents to functions of the form $y = f(x)$ and $z = f(x, y)$. In the case of $y = f(x)$ we can imagine a tangent *line* and for $z = f(x, y)$ we can imagine a tangent *plane*.

In both cases, the mathematical object that we create is a linear approximation of the function that does a pretty good job of approximating function values in a neighborhood of the point of tangency.



Suppose we now consider a function $v = f(x, y, z)$. We can follow the analogy of the tangent line and tangent plane formulas to create an analogous formula for a higher dimensional function. But how can you imagine what that analogous formula actually represents? Answering this question is the subject of today's recitation.

Higher Dimensional Linear Approximations

For the function $y = f(x)$, the linear approximation to the function at $x = x_0$ is the equation of the tangent line:

$$y - f(x_0) = f'(x_0)\cdot(x - x_0)$$

or:

$$y = f(x_0) + f'(x_0)\cdot(x - x_0)\ \ldots\ (1)$$

For the function $z = f(x, y)$, the linear approximation to the function at $(x, y) = (x_0, y_0)$ is the equation of the tangent plane:

$$\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0) - (z - f(x_0,y_0)) = 0$$

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or:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) \ \ldots \ (2).$$

1. Consider the function $v = g(x, y, z)$. Study the patterns of Equations (1) and (2) (above) and write a general formula for the linear approximation to the function at the point $(x, y, z) = (x_0, y_0, z_0)$.

$$v = g(x_0,y_0,z_0) + \frac{\partial g}{\partial x}\cdot(x-x_0) + \frac{\partial g}{\partial y}\cdot(y-y_0)$$

$$+ \frac{\partial g}{\partial z}(z-z_0)$$

All partial derivatives evaluated at (x_0, y_0, z_0).

2. The linear approximation of the function $v = q(x, y, z) = x\cdot y\cdot z$ at the point $(x, y, z) = (1, 2, 3)$ is:

$$v = 6 + 6\cdot(x-1) + 3\cdot(y-2) + 2\cdot(z-3).$$

Plug the function $q(x, y, z)$ and the point $(x, y, z) = (1, 2, 3)$ into your general formula from Question 1 to see if your general formula is correct. If it isn't correct, try to fix your general formula.

$$q(1,2,3) = (1)(2)(3) = 6.$$

$$\frac{\partial q}{\partial x} = y\cdot z \qquad \frac{\partial q}{\partial x}(1,2,3) = (2)(3) = 6$$

$$\frac{\partial q}{\partial y} = x\cdot z \qquad \frac{\partial q}{\partial y}(1,2,3) = (1)(3) = 3$$

$$\frac{\partial q}{\partial z} = x\cdot y \qquad \frac{\partial q}{\partial z}(1,2,3) = (1)(2) = 2.$$

Linear approximation:

$$v = 6 + 6\cdot(x-1) + 3\cdot(y-2) + 2\cdot(z-3).$$

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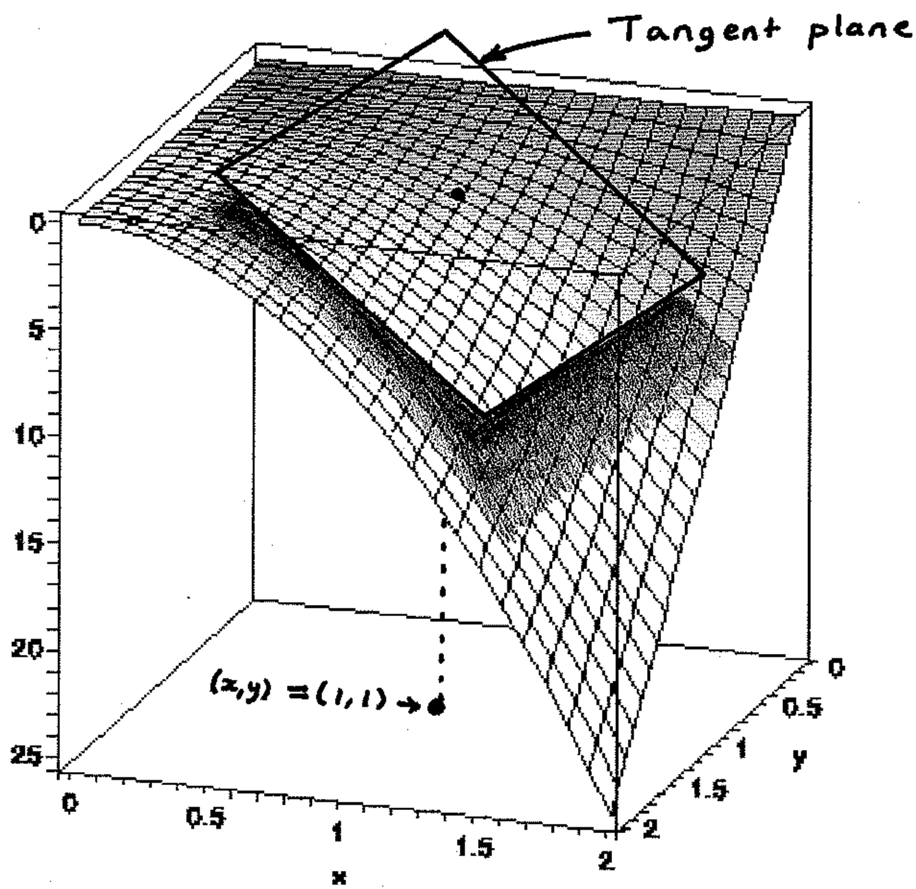
Visualizing the Linear Approximation for $f(x, y)$

As we have noted, one way to visualize the linear approximation for a function of the form $z = f(x, y)$ is to imagine a tangent plane.

In this part of the recitation we will learn another way to imagine the linear approximation, a way that we will generalize to $v = g(x, y, z)$ in the last part of the recitation.

3. The graph of the function $z = f(x, y) = \pi x^2 y$ is shown below. As accurately as you can, draw the tangent plane to the point $(x, y) = (1, 1)$ over the top of the graph.



(Note that to make the tangent plane a little easier to draw, the graph has been drawn in an unusual way with the positive z-axis extending down the page.)

Based on your sketch, do you think that the z-value of the tangent plane at a point (x, y) that is near $(1, 1)$ will be larger or smaller than the value of $z = f(x, y)$? Remember that the graph is drawn with z-values increasing as you go **down** the page.

The z-values of the tangent plane will be smaller than the corresponding values of $z = f(x, y)$.

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4. Calculate the equation of the tangent plane of $z = f(x, y) = \pi x^2 y$ when the point of tangency is the point $(x, y) = (1, 1)$.

$$f(1,1) = \pi(1)^2(1) = \pi$$

$$\frac{\partial f}{\partial x} = 2\pi x y \qquad \frac{\partial f}{\partial x}(1,1) = 2\pi(1)(1) = 2\pi$$

$$\frac{\partial f}{\partial y} = \pi x^2 \qquad \frac{\partial f}{\partial y}(1,1) = \pi(1)^2 = \pi.$$

Equation of Linear Approximation:

$$z = \pi + 2\pi\cdot(x-1) + \pi(y-1).$$

5. Use the equation that you obtained for the tangent plane to complete the values in the table given below. These values are all estimates of the value of $f(x, y)$ using the tangent plane to approximate the value of $f(x, y)$ near $(x, y) = (1, 1)$.

x	y	z
1.1	1	$1.2\pi \approx 3.769911184$
1	1.1	$1.1\pi \approx 3.455751919$

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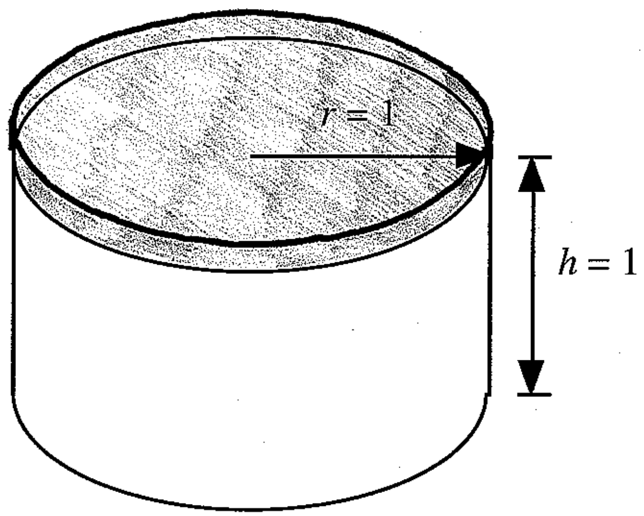
Calculus Experiments with Play Doh

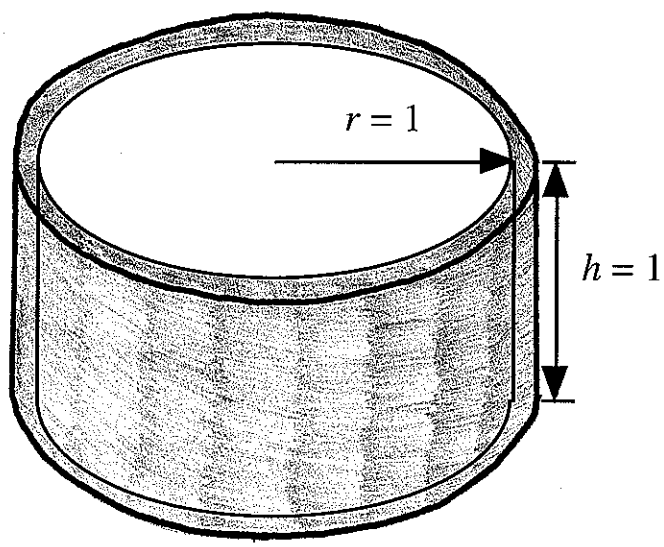
6. If we replace the letters x and y in the formula for $f(x, y)$ by the letters r and h, then the formula for $f(x, y)$ resembles the formula for the volume of a cylinder:

$$z = f(r, h) = \pi \cdot r^2 \cdot h.$$

Imagine that you have a cylinder with radius 1 and height 1 (as shown in the diagram below). Modify this diagram to show how the appearance of the cylinder would be different if the height of the cylinder was increased from $h = 1$ to $h = 1.1$. Calculate the new volume.



7. Now, imagine that we have the cylinder with $r = 1$ and $h = 1$ again, and that this time the cylinder is changed by increasing the radius from $r = 1$ to $r = 1.1$. Modify the diagram (given below) to show how the appearance of the cylinder is different.



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8. Use the method of calculation described below to calculate the volume of Play Doh in a tub of Play Doh.

(a) Get a tub of Play-Doh from your recitation instructor.

(b) Pack the Play-Doh around the curved side of the cylindrical tub so that it forms an even layer.

(c) Use a knife to slit the Play Doh vertically down one side.

(d) Unroll the Play Doh from the cylinder so that it is flat and forms a rectangular slab.

(e) Use a ruler to measure the length, width and thickness of the slab of Play Doh and multiply them together to get the volume.

Quantity	Value (cm or cm^3)
Length	
Width	
Thickness	
Volume	

9. How could you have estimated the length and the width of the Play Doh slab using measurements taken from the plastic Play Doh tub?

The width is the height of the Play Doh tub.

The length is the circumference of the Play Doh tub.

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10. Adapt your answers to Questions 8 and 9 to estimate the *extra* volume that you drew on the diagram from Question 7. Add this extra volume to the original volume to get an estimate for the *total* volume.

If you just increase the height from $h=1$ to $h=1.1$, the total volume is:

$$\text{Volume} = \pi(1)^2 \cdot (1.1) = 1.1\pi$$

$$\approx 3.455751919.$$

If you just increase the radius from $r=1$ to $r=1.1$, the total volume of the cylinder is approximately:

$$\text{Volume} = \pi + 2\pi(1)(1)(0.1) = 1.2\pi$$

$$\approx 3.769911184$$

11. Compare the numbers that you obtained in Question 6 and Question 10 to the numbers that you recorded in the table in Question 5. What do you notice?

The numbers are the same.

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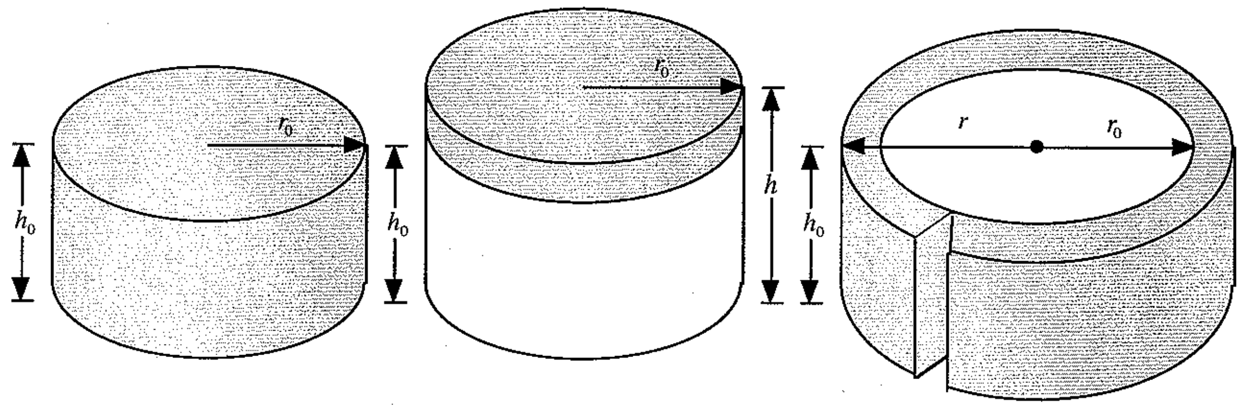
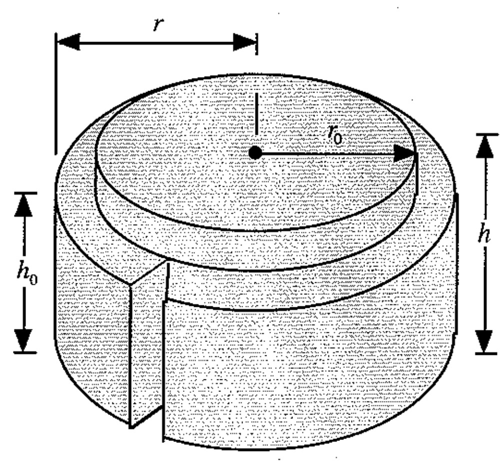
Visualizing the Linear Approximation for $f(x, y, z)$

You will probably have noticed a correspondence between the numbers when you calculated Question 5 and Question 11. This is no accident, and this correspondence provides us with a way to visualize the meaning of the terms in a linear approximation. The relationship between the volumes that you calculated and the terms in the linear approximation for $f(r, h) = \pi r^2 \cdot h$ is illustrated below.



$f(r_0, h_0) = \pi \cdot r_0^2 \cdot h_0$ $\quad\quad$ $\frac{\partial f}{\partial h} \cdot (h - h_0) = \pi \cdot r_0^2 \cdot (h - h_0)$ $\quad\quad$ $\frac{\partial f}{\partial r} \cdot (r - r_0) = 2 \cdot \pi \cdot r_0 \cdot h \cdot (r - r_0)$

The total volume that is equal to the value of the linear approximation for $f(r, h) = \pi r^2 \cdot h$ evaluated at the point (r, h) is the following.



Note that this volume is not exactly the same as the volume of the cylinder with radius r and height h. The volume of the cylinder with radius r and height h includes the volume of the small wedge in the side and the "ring" around the cap at the top. However, if the differences $r - r_0$ and $h - h_0$ are both small then the volumes of the wedge and the ring are also small, and the volume of the shape shown above is a close approximation of the value of $f(r, h)$.

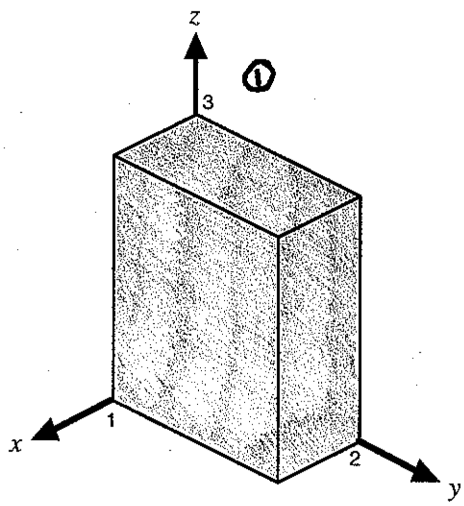
12. Go back and read your answer to Question 3 of this recitation. How could you have predicted that whatever shape that you get for the linear approximation to $f(r, h)$ will have a volume that is less than the volume of the cylinder of radius r and height h?

So, one way to visualize the linear approximation for a function is to imagine $f(x_0, y_0, z_0)$ as a volume, and the terms of the linear approximation as the additions to that volume when the length of each side is increased slightly, from x_0 to x, from y_0 to y and from z_0 to z. In this last part of the recitation we will adapt this insight to develop a way to visualize the meaning of the linear approximation to a higher dimensional function such as $f(x, y, z)$.

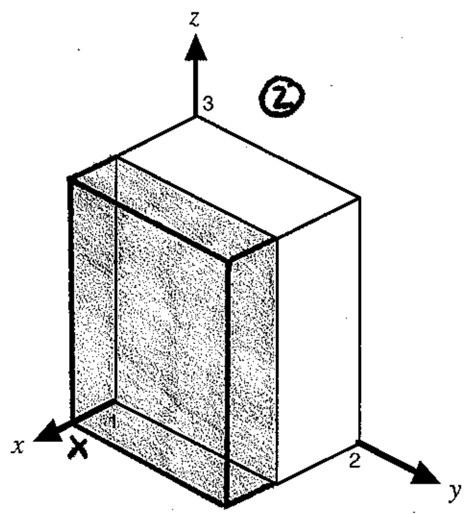
13. Consider the higher dimensional function: $v = q(x,y,z) = x \cdot y \cdot z$. In Question 2, you verified that the linear approximation of this function at the point $(x, y, z) = (1, 2, 3)$ is:

$$v = 6 + 6\cdot(x-1) + 3\cdot(y-2) + 2\cdot(z-3).$$

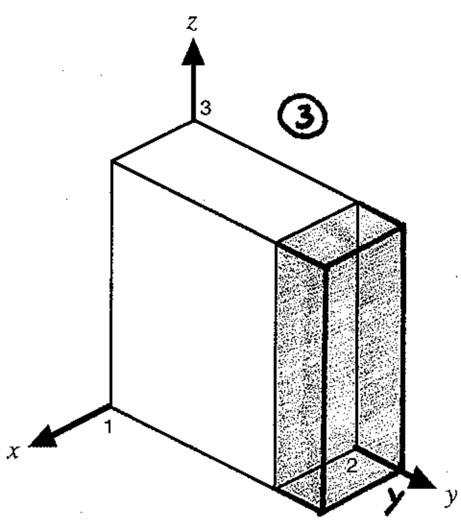
Use the diagrams shown below to sketch volumes that correspond to each of the terms of the linear approximation of $q(x, y, z)$.



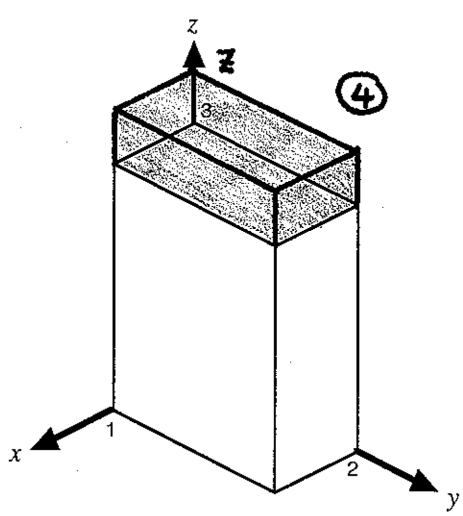
$q(1,2,3) = 6$



$$\left(\left.\frac{\partial q}{\partial x}\right|_{(x,y,z)=(1,2,3)}\right)\cdot(x-1) = 6\cdot(x-1)$$



$$\left(\left.\frac{\partial q}{\partial y}\right|_{(x,y,z)=(1,2,3)}\right)\cdot(y-2) = 3\cdot(y-2)$$



$$\left(\left.\frac{\partial q}{\partial z}\right|_{(x,y,z)=(1,2,3)}\right)\cdot(z-3) = 2\cdot(z-3)$$

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14. What are the areas of the three faces of the rectangular boxes from Question 13? How are these areas connected with the values 6, 3 and 2 that appear in the formula for the linear approximation? How do the quantities $(x - 1)$, $(y - 2)$ and $(z - 3)$ from the formula for the linear approximation appear in the diagrams that you drew in Question 13?

The area of the face parallel to the yz plane is 6.
The area of the face parallel to the xz plane is 3.
The area of the face parallel to the xy plane is 2.

These are the values of the partial derivatives of $q(x, y, z)$ evaluated at the point (x, y, z).

$(x-1)$ is the thickness of the gray volume in 2.
$(y-2)$ is the thickness of the gray volume in 3.
$(z-3)$ is the thickness of the gray volume in 4.

15. Use the diagram shown below to draw a picture that shows the value of the linear approximation of $q(x, y, z)$ evaluated at a point (x, y, z) near (1, 2, 3) visualized as a volume.