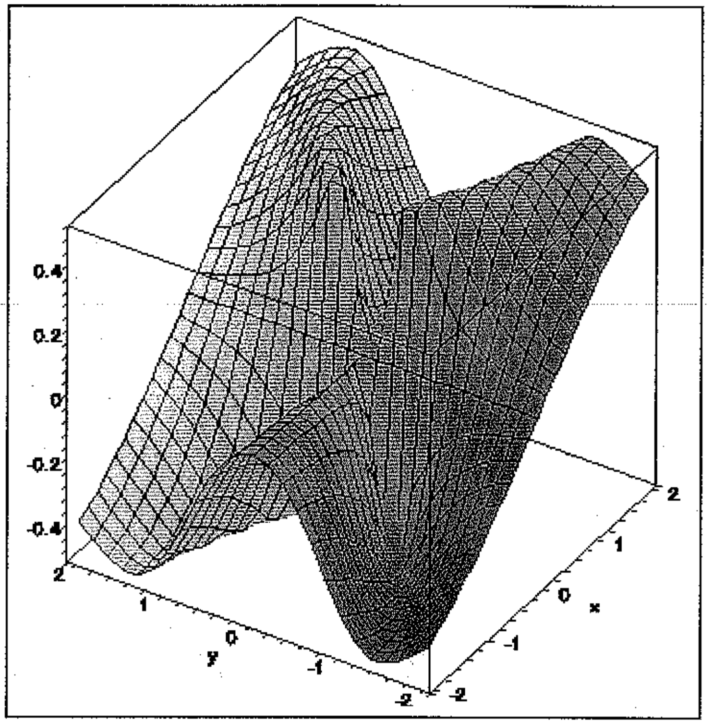
SOLUTIONS

Recitation Handout 6: Limits in Two Dimensions



As we have discussed in lecture, investigating the behavior of functions with two variables, $f(x, y)$, can be more difficult than functions of only one variable, $f(x)$. This is particularly the case when using limits to investigate the behavior of functions when conventional arithmetic breaks down.

The purpose of this recitation is to equip with knowledge of a range of examples with three goals:

(a) Indicate some tools and strategies that you can use to help you to decide when a limit does and does not exist.

(b) Practice the calculations that you need to carry out to show that a limit does not exist.

Showing that a Limit Does Not Exist

It is often easier to show that a limit does not exist. This is because the output (or z-value) of the function $f(x, y)$ must approach to same value regardless of how the point in the xy-plane (a, b) is approached. As there are so many ways to approach any given point in the xy-plane, it is often possible to find two that both approach the point $(x, y) = (a, b)$ but along which the function $f(x, y)$ approaches different values.

To show that a limit does not exist:

(a) Find two curves in the xy-plane, say $y = f_1(x)$ and $y = f_2(x)$ that both include the point $(x, y) = (a, b)$.

(b) Use the two paths to substitute for y in the function $f(x, y)$ and in each case take the limit as $x \to a$.

(c) If you get different values of the limit as $x \to a$ then you have demonstrated that the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does not exist.

SOLUTIONS

Example

1. Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Path 1:

The x-axis $y = 0$. On the x-axis:

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2}{x^2} = 1, \quad x \neq 0. \qquad \frac{x^2-y^2}{x^2+y^2} \to 1$$ along the x-axis.

Limit along Path 1 as $x \to 0$:

Along the x-axis, $\dfrac{x^2-y^2}{x^2+y^2} \to 1$ as $(x,y) \to (0,0)$

along the x-axis.

Path 2:

The y-axis $x = 0$. On the y-axis:

$$\frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2} = -1, \quad y \neq 0.$$

Limit along Path 2 as $x \to 0$:

Along the y-axis, $\dfrac{x^2-y^2}{x^2+y^2} \to -1$ as $(x,y) \to (0,0)$

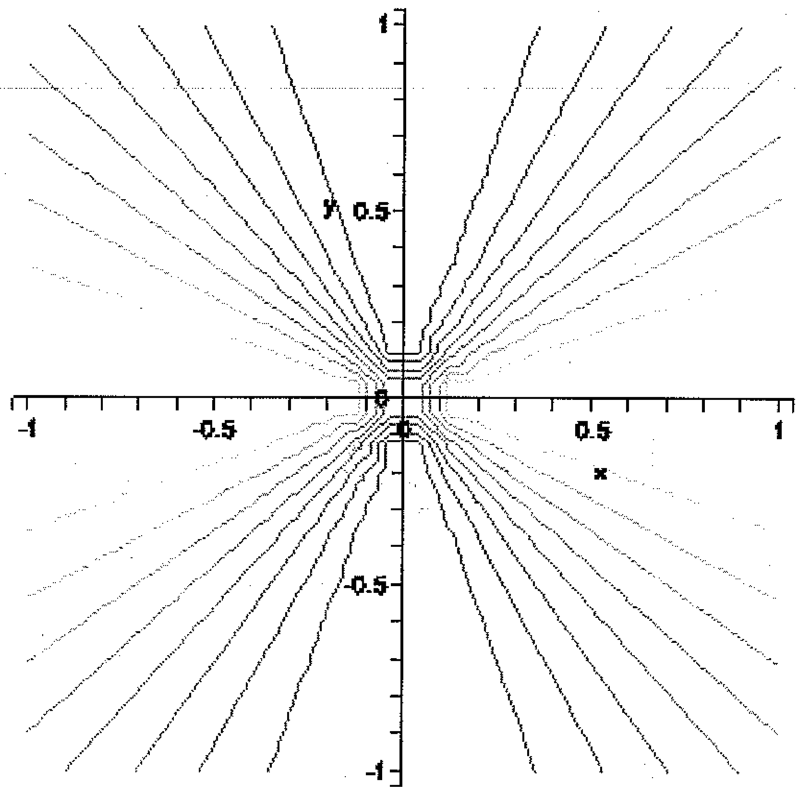
along the y-axis.

SOLUTIONS

Tool 1: The Contour Plot

A contour plot can sometimes reveal when a function does not have a limit at a particular point. The signature of a limit not existing in a contour plot is contour lines that cross or converge at the point $(x, y) = (a, b)$ you are interested in.

The contour plot of $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is shown below. Although this computer-generated graph does not show it clearly, the contour lines converge on the point (0, 0) at the center of the graph.



Setting $z = f(x, y) = k$ shows the convergence of the contour lines.

$$k = \frac{x^2 - y^2}{x^2 + y^2}$$

$$k \cdot \left(x^2 + y^2\right) = x^2 - y^2$$

$$(1 + k) \cdot y^2 = (1 - k) \cdot x^2$$

$$y = \pm\sqrt{\frac{1-k}{1+k}} \cdot x.$$

The contours are straight lines that converge on the point (0, 0). The convergence of the contour lines at (0, 0) shows that the limit of $f(x, y)$ does not exist there.

SOLUTIONS

Example

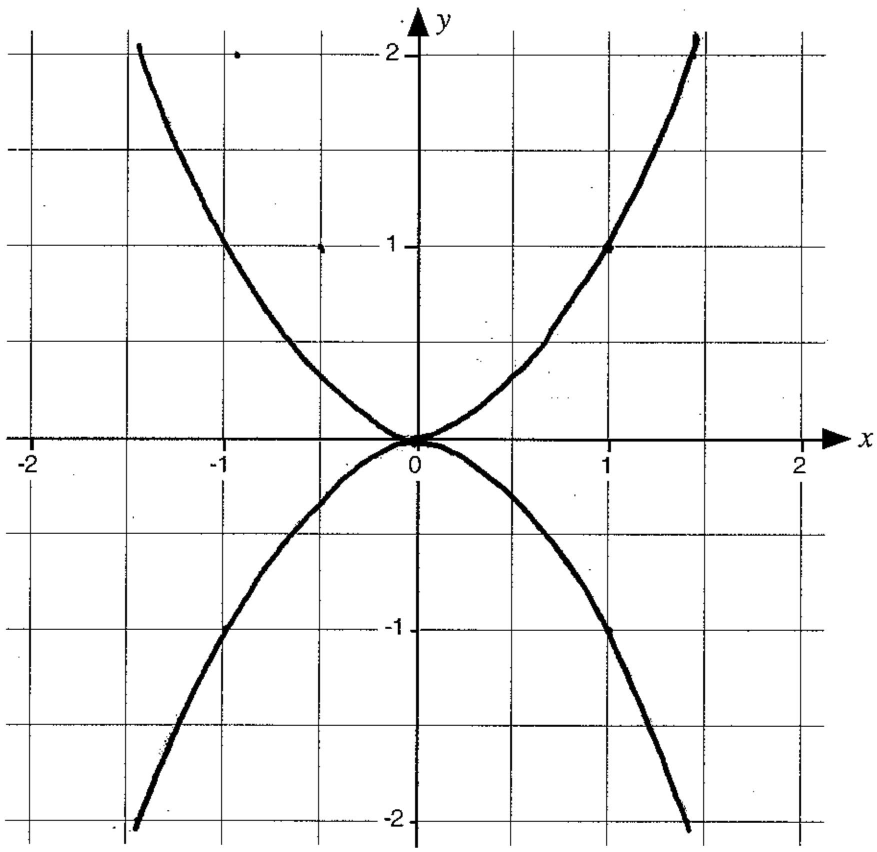
2. Use the axes provided below to sketch a contour plot for the function:

$$g(x,y)=\frac{2x^2y}{x^4+y^2}.$$

$k(x^4+y^2)=2x^2y$ So: $ky^2-2x^2y+kx^4=0$

$$y=\frac{2x^2\pm\sqrt{4x^4-4k^2x^4}}{2k}=\begin{cases}\dfrac{1+\sqrt{1-k^2}}{k}\cdot x^2\\[2ex]\dfrac{1-\sqrt{1-k^2}}{k}\cdot x^2\end{cases}$$

Contours are parabolas through $(0,0)$



3. How can you use your contour plot to determine whether the limit:

$$\lim_{(x,y)\to(0,0)}\frac{2x^2y}{x^4+y^2}$$

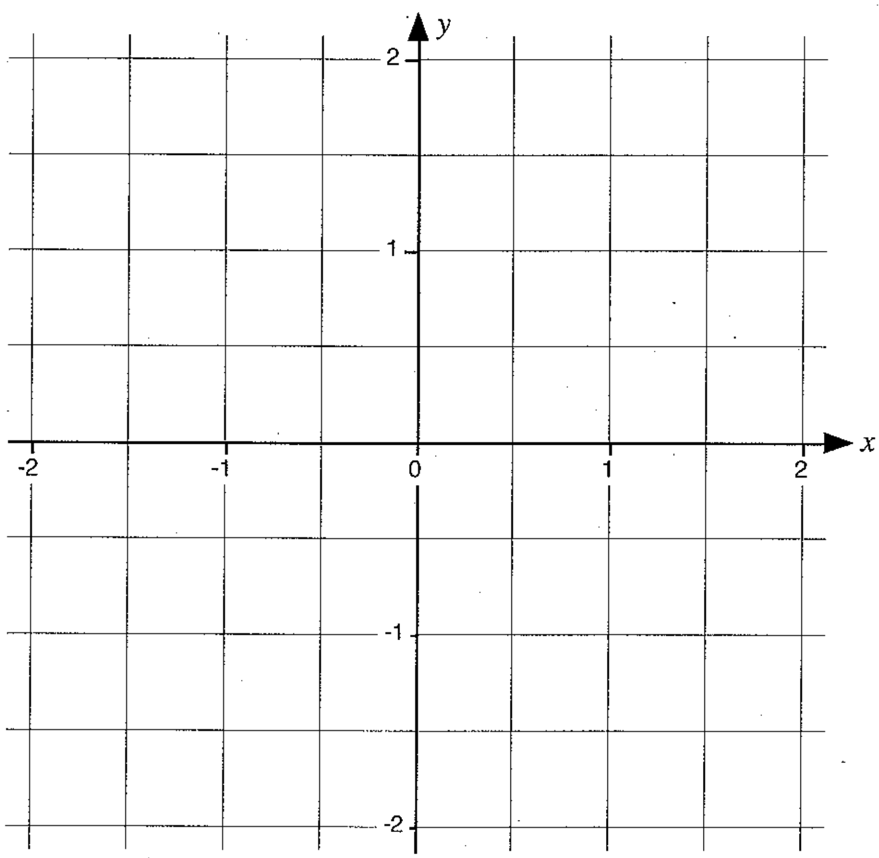
exists or not?

You can examine the plot to see what the contours do near $(x,y)=(0,0)$. If any of the contours that correspond to different values of k converge at $(0,0)$, the limit will not exist there.

SOLUTIONS

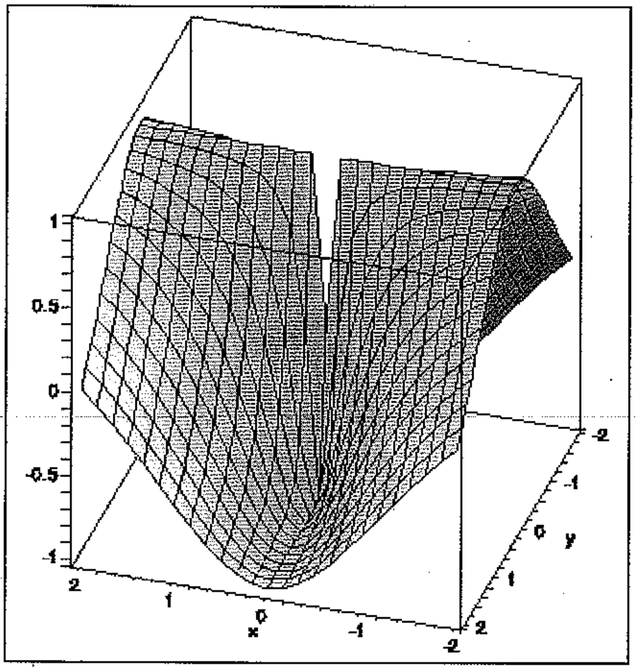
4. Find the value of the limit $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ or show that the limit does not exist.

The limit does not exist as contours corresponding to different values of k converge at $(x,y) = (0,0)$.



SOLUTIONS

Tool 2: A Table



A table can provide some clues on whether a limit exists or not. Unlike a contour plot, a table cannot decisively determine whether a limit exists or not. However, contour plots are sometimes difficult to create by hand, and computer-generated contour plots can be deceptive. (For example, the computer-generated contour plot shown earlier is not accurate near the point (0, 0).)

The graph of the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ is shown in the diagram and some of the z-values of this function are shown in the table (below). The entries in the table are the values of $f(x, y)$ with the values of x and y shown at the edges of the table plugged into the function.

	x = −1.0	x = −0.5	x = −0.2	x = 0	x = 0.2	x = 0.5	x = 1.0
y = 1.0	0.000	−0.600	−0.923	−1.000	−0.923	−0.600	0.000
y = 0.5	0.600	0.000	−0.724	−1.000	−0.724	0.000	0.600
y = 0.2	0.923	0.724	0.000	−1.000	0.000	0.724	0.923
y = 0	1.000	1.000	1.000		1.000	1.000	1.000
y = −0.2	0.923	0.724	0.000	−1.000	0.000	0.724	0.923
y = −0.5	0.600	0.000	−0.724	−1.000	−0.724	0.000	0.600
y = −1.0	0.000	−0.600	−0.923	−1.000	−0.923	−0.600	0.000

This table suggests that the limit of $f(x, y)$ does not exist as $(x, y) \to (0, 0)$ because as you work your way into the center of the table along different rows or columns, the values that you see in the table seem to converge to different values.

For example, if approach (0, 0) along the center row, the value of $f(x, y)$ seems to be fixed at 1.000. However, if you approach (0, 0) along the middle column, the value of $f(x, y)$ seems to be fixed at −1.000. This suggests (but does not prove) that the limit does not exist.

SOLUTIONS

Example

6. Complete the table given below using the function:

$$p(x,y)=\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}.$$

	x = −1.0	x = −0.5	x = −0.2	x = 0	x = 0.2	x = 0.5	x = 1.0
y = 1.0	2.732	2.500	2.428	2.414	2.428	2.500	2.732
y = 0.5	2.500	2.225	2.136	2.118	2.136	2.225	2.500
y = 0.2	2.428	2.136	2.039	2.019	2.039	2.136	2.428
y = 0	2.414	2.118	2.019		2.019	2.118	2.414
y = −0.2	2.428	2.136	2.039	2.019	2.039	2.136	2.428
y = −0.5	2.500	2.225	2.136	2.118	2.136	2.225	2.500
y = −1.0	2.732	2.500	2.428	2.414	2.428	2.500	2.732

7. Based on the table that you completed in Question 6, do you think that the limit:

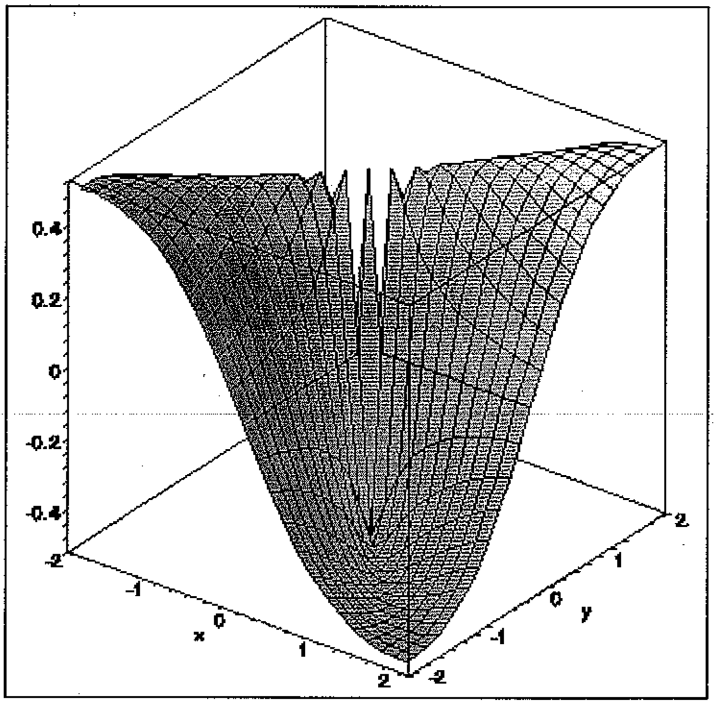
$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

exists or not? Find the value or the limit or show that it does not exist.

Based on the entries in the table, I would guess that the limit does exist. Plugging smaller values of x and y into p(x, y) suggests:

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}=2.$$

Tool 3: The Paths $y - b = m \cdot (x - a)$



Building a table can help to shed some light on whether a limit exists or not. The main problem with building a table is that you can only include a limited number of cells in the table. As such, if you choose unrepresentative values of x and y for your tables, you can get a distorted view of how the function behaves.

An algebraic tool that can to shed some light on the existence of a limit is to reduce the problem from a two dimensional limit to a family of one-dimensional limits. Like the table, this approach cannot always definitely answer the question of whether a limit exists or not.

To test the existence of a limit such as $\lim\limits_{(x,y)\to(a,b)} g(x,y)$:

(a) Using the numerical values of a and b and an unspecified constant m, write down the formula for the linear function:

$$y = m \cdot (x - a) + b.$$

(b) Use this linear function to substitute for y in the formula for $g(x, y)$ and simplify this function as much as possible.

(c) Use your knowledge of limits in one dimension to calculate the limit of your expression as $x \to a$.

(d) If the limit you calculate depends on m, then the two dimensional limit of $g(x, y)$ does not exist as $(x, y) \to (a, b)$.

(e) If the limit you calculate does not depend on m then it is possible that the two-dimensional limit may exist, but this does not prove that the two-dimensional limit exists.

SOLUTIONS

Example

8. By making the substitution $y = m \cdot x$, show that the limit:

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$$

does not exist.

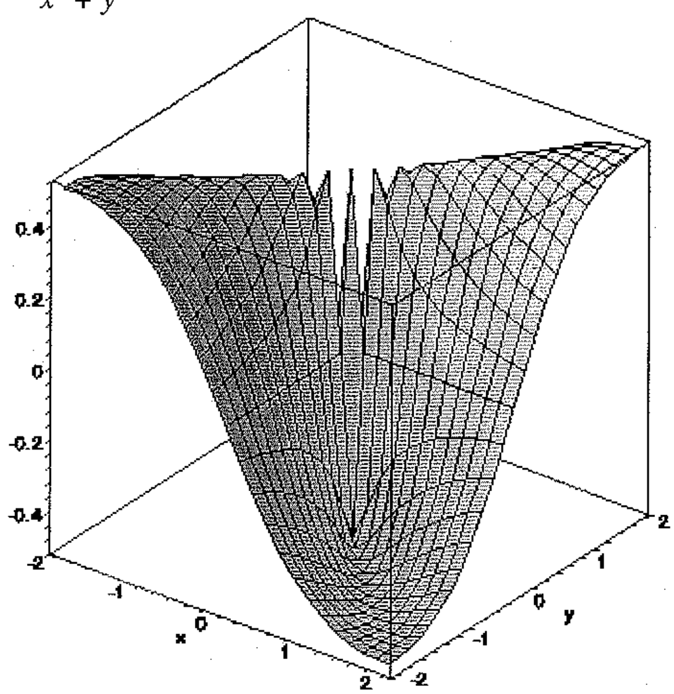
Substituting $y = m \cdot x$ gives:

$$\frac{x \cdot y}{x^2 + y^2} = \frac{x \cdot m \cdot x}{x^2 + (mx)^2} = \frac{m}{1 + m^2}$$

Different values of m give different values of $\frac{m}{1 + m^2}$, so $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$

does not exist.

The graph of $z = \frac{xy}{x^2 + y^2}$ is shown below.



SOLUTIONS

9. Use the substitution $y = m{\cdot}x$ to determine whether or not the limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

exists or not. Find the value of the limit or show that it does not exist.

Using the substitution $y = m\cdot x$ gives:

$$\frac{x\cdot y^2}{x^2+y^4} = \frac{x\cdot m^2\cdot x^2}{x^2+m^4\cdot x^4} = \frac{m^2\cdot x}{1+m^4 x^2}, \quad x \neq 0.$$

$$\lim_{x\to 0} \frac{m^2\cdot x}{1+m^2\cdot x^2} = 0.$$

This result does not involve m. This does not prove that the limit exists, but this result does not immediately show that the limit does not exist.

10. Try the substitution $x = y^2$ in the limit from Question 9. Remember that *any* path in the *xy*-plane means *ANY* path, not just the ones that have the equation $y = = m{\cdot}x$.

Using the substitution $x = y^2$ gives:

$$\frac{x\cdot y^2}{x^2+y^4} = \frac{x\cdot x}{x^2+x^2} = \frac{1}{2}.$$

This shows that as $(x,y)\to(0,0)$ along $x = y^2$,

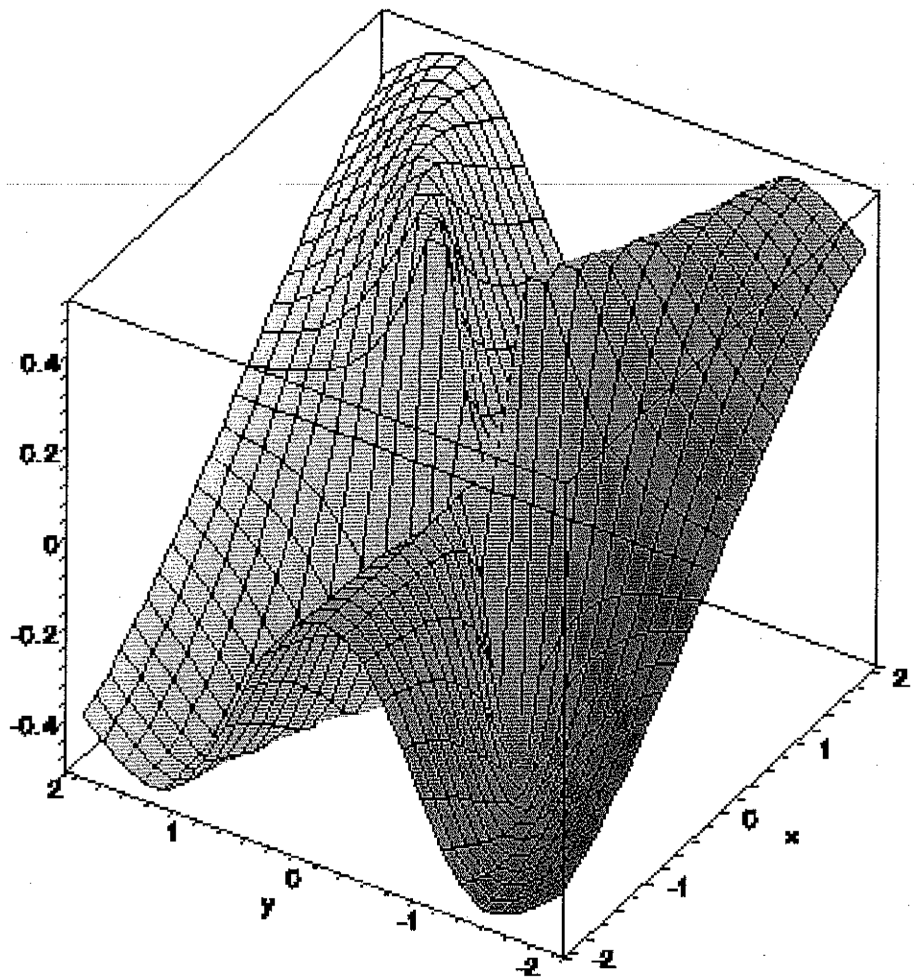
$\dfrac{xy^2}{x^2+y^4} \to \dfrac{1}{2}$. Since this is not equal to zero,

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$ does not exist.

SOLUTIONS

Limitations of Tool 3

The graph of $z = \dfrac{xy^2}{x^2 + y^4}$ is shown below. Note the "groove" that lies along the path $x = y^2$. Along this groove, the z-value of the graph approaches $z = 0$ despite the fact that $z \to 0.5$ along each straight-line path that passes through (0, 0).



This surface illustrates why Tool 3 cannot prove that a limit exists. Tool 3 only investigates the behavior of the function along straight lines that converge on the point (a, b), and there are many, many other paths in the xy-plane that converge on the point (a, b). The behavior of the function may be different along these other paths, and it only takes one path to prevent a limit from existing.