SOLUTIONS

Recitation Handout 4: Proving Vector Identities

The purpose of this recitation is to give you some practice at establishing vector identities (or formulas), especially those that include dot or cross products.

When you attempt your proofs, you may make basic assumptions about the vectors. For example, in a three-dimensional situation you can assume that the vectors you are working with are three-dimensional vectors. However, it is not sufficient to choose a few specific vectors and show that the identity is verified for them.

1. Consider two three-dimensional vectors $\vec{a}$ and $\vec{b}$. Show that:

$$\left|\vec{a}\times\vec{b}\right|^2 = \left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2 - \left(\vec{a}\bullet\vec{b}\right)^2.$$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$. Let $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\vec{a}\times\vec{b} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

So:

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$\text{RHS} = |\vec{a}|^2\cdot|\vec{b}|^2 - (\vec{a}\bullet\vec{b})^2$$

$$= (a_1^2 + a_2^2 + a_3^2)\cdot(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2$$
$$+ a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 - a_1^2b_1^2 - a_1b_1a_2b_2 - a_1b_1a_3b_3$$
$$- a_2b_2a_1b_1 - a_2^2b_2^2 - a_2b_2a_3b_3 - a_3b_3a_1b_1 - a_3b_3a_2b_2$$
$$- a_3^2b_3^2$$

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$$\begin{aligned} \text{RHS} = \ & a_2^2 b_3^2 - 2 a_2 b_3 a_3 b_2 + a_3^2 b_2^2 \\ & + a_3^2 b_1^2 - 2 a_3 b_1 a_1 b_3 + a_1^2 b_3^2 \\ & + a_1^2 b_2^2 - 2 a_1 b_2 a_2 b_1 + a_2^2 b_1^2 \\ = \ & (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 \\ & + (a_1 b_2 - a_2 b_1)^2 \\ = \ & |\vec{a} \times \vec{b}|^2 \\ = \ & \text{LHS}. \end{aligned}$$

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2. Consider three three-dimensional vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Show that:

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=(\vec{a}\bullet\vec{c})\vec{b}-\left(\vec{b}\bullet\vec{c}\right)\vec{a}.$$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and $\vec{c} = \langle c_1, c_2, c_3 \rangle$.

Then:

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \rangle$$

so that:

$$(\vec{a} \times \vec{b}) \times \vec{c} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\begin{aligned} = \langle\, & c_3(a_3b_1 - a_1b_3) - c_2(a_1b_2 - a_2b_1), \\ & c_1(a_1b_2 - a_2b_1) - c_3(a_2b_3 - a_3b_2), \\ & c_2(a_2b_3 - a_3b_2) - c_1(a_3b_1 - a_1b_3) \,\rangle \end{aligned}$$

Now, $\vec{a} \bullet \vec{c} = a_1c_1 + a_2c_2 + a_3c_3$, and

$\vec{b} \bullet \vec{c} = b_1c_1 + b_2c_2 + b_3c_3$, so that:

$$
\begin{aligned}
\text{RHS} &= (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a} \\
&= (a_1 c_1 + a_2 c_2 + a_3 c_3) \langle b_1, b_2, b_3 \rangle \\
&\quad - (b_1 c_1 + b_2 c_2 + b_3 c_3) \langle a_1, a_2, a_3 \rangle \\
&= \langle (a_1 c_1 + a_2 c_2 + a_3 c_3) b_1 - (b_1 c_1 + b_2 c_2 + b_3 c_3) a_1 , \\
&\qquad (a_1 c_1 + a_2 c_2 + a_3 c_3) b_2 - (b_1 c_1 + b_2 c_2 + b_3 c_3) a_2 , \\
&\qquad (a_1 c_1 + a_2 c_2 + a_3 c_3) b_3 - (b_1 c_1 + b_2 c_2 + b_3 c_3) a_3 \rangle \\
&= \langle a_2 c_2 b_1 + a_3 c_3 b_1 - b_2 c_2 a_1 - b_3 c_3 a_1 , \\
&\qquad a_1 c_1 b_2 + a_3 c_3 b_2 - b_1 c_1 a_2 - b_3 c_3 a_2 , \\
&\qquad a_1 c_1 b_3 + a_2 c_2 b_3 - b_1 c_1 a_3 - b_2 c_2 a_3 \rangle \\
&= \langle c_3 (a_3 b_1 - a_1 b_3) - c_2 (a_1 b_2 - a_2 b_1) , \\
&\qquad c_1 (a_1 b_2 - a_2 b_1) - c_3 (a_3 b_2 - b_3 a_2) , \\
&\qquad c_2 (a_2 b_3 - b_2 a_3) - c_1 (a_3 b_1 - a_1 b_3) \rangle \\
&= (\vec{a} \times \vec{b}) \times \vec{c} \\
&= \text{LHS.}
\end{aligned}
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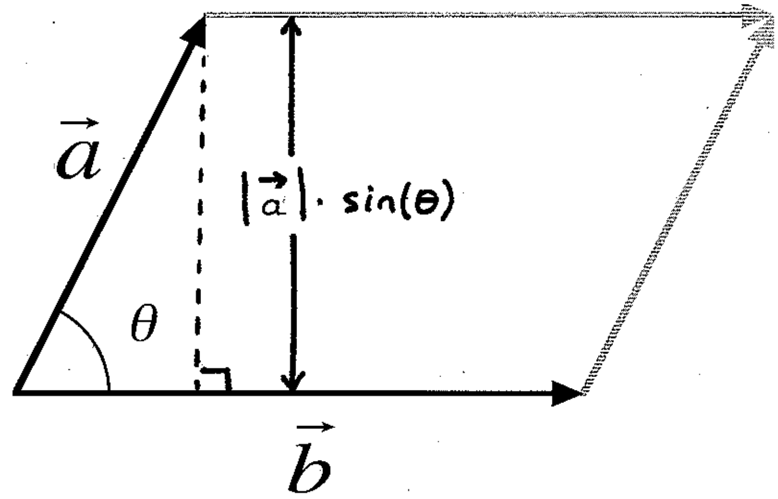
3. Consider the parallelogram formed by two three-dimensional vectors, $\vec{a}$ and $\vec{b}$, as shown below.



Show that the area of this parallelogram is equal to $|\vec{a} \times \vec{b}|$.

On one hand the area of the parallelogram is the area of the rectangle with length $|\vec{b}|$ and width $|\vec{a}| \cdot \sin(\theta)$. The area is:

$$|\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta).$$

Next we will show that $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Then $$\vec{a} \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \langle a_2 b_3 - a_3 b_2 , a_3 b_1 - a_1 b_3 , a_1 b_2 - b_1 a_2 \rangle$$

So that: $|\vec{a} \times \vec{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - b_1 a_2)^2.$

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Now, $\dfrac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \cos(\theta)$ so that:

$$\vec{a} \bullet \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\theta).$$

Next, note that:

$$|\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \bullet \vec{b})^2 = |\vec{a} \times \vec{b}|^2$$

(proved earlier) so that:

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 - \left(|\vec{a}| \cdot |\vec{b}| \cdot \cos(\theta) \right)^2$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot \left(1 - \cos^2(\theta) \right)$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot \sin^2(\theta).$$

Now, when $0 < \theta < \pi$,

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)$$

because when $0 < \theta < \pi$, $\sin(\theta) > 0$.

This shows that (in addition to being equal to the area of the parallelogram), $|\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)$ is also equal to $|\vec{a} \times \vec{b}|$. This means that the area of the parallelogram is also equal to $|\vec{a} \times \vec{b}|$.

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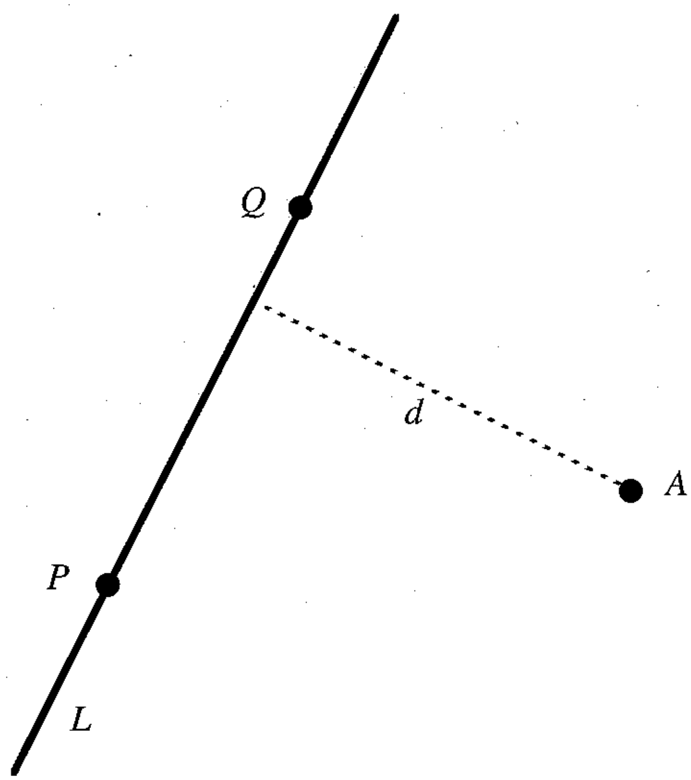
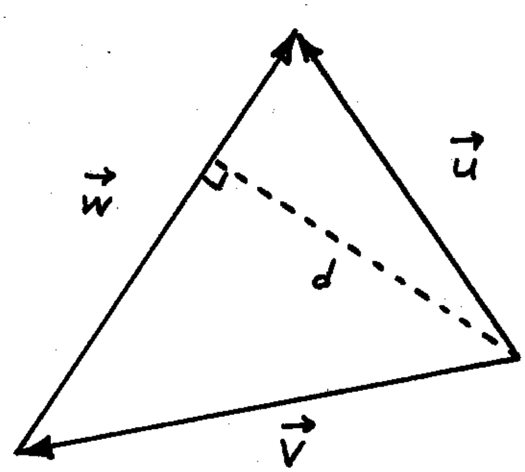
4. Suppose that P and Q are distinct points on a line, L, in three-dimensional space. Let A be another point in three-dimensional space that does not lies on L. (For example, see the diagram given below.) The points A, P and Q form a triangle. The height of this triangle, d, gives the minimum distance from the line L to the point A.



Let $\vec{u}$ denote the vector from point A to point P, $\vec{v}$ denote the vector from point A to point Q and $\vec{w}$ denote the vector from point P to point Q. Show that:

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{w}|}.$$

We will begin by drawing the three vectors.



Now, on one hand the area of the triangle formed by the three vectors is:

$$\text{Area} = \tfrac{1}{2}\,(\text{Base})\,(\text{Height})$$
$$= \tfrac{1}{2} \cdot |\vec{w}| \cdot d.$$

On the other hand, the area of the triangle is half of the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$. By the previous result,

$$\text{Area} = \tfrac{1}{2}\,|\vec{u} \times \vec{v}|.$$

Equating the two expressions for area and simplifying gives:

$$\tfrac{1}{2} \cdot |\vec{w}| \cdot d = \tfrac{1}{2}\,|\vec{u} \times \vec{v}|$$

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{w}|}$$

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5. The determinant of a three-by-three matrix is a number given by the formula:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

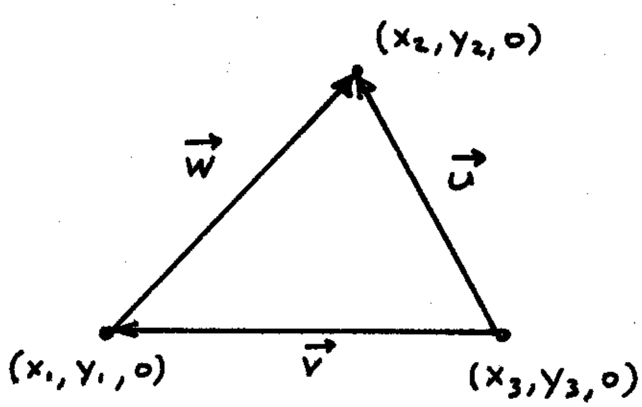
Consider the triangle with vertices located at $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$. Show that the area of this triangle is equal to one half times the absolute value of the determinant of the matrix:

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}.$$

$$\text{LHS} = \frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$= \frac{1}{2}\left(x_2 y_3 + x_1 y_2 + y_1 x_3 - y_1 x_2 - y_2 x_3 - x_1 y_3 \right)$$

Using the same notation as the previous problem, we can write:



$$\vec{u} = \langle x_2 - x_3,\ y_2 - y_3,\ 0 \rangle$$

$$\vec{v} = \langle x_1 - x_3,\ y_1 - y_3,\ 0 \rangle$$

$$\vec{w} = \langle x_2 - x_1,\ y_2 - y_1,\ 0 \rangle$$

To compute the area of this triangle, we can plug the above vectors into:

$$\text{Area} = \frac{1}{2} \left| \vec{u} \times \vec{v} \right|.$$

Now,

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 \end{bmatrix}$$

$$= \langle 0,\ 0,\ (x_2 - x_3)(y_1 - y_3) - (y_2 - y_3)(x_1 - x_3) \rangle$$

$$= \langle 0,\ 0,\ x_2 y_1 - y_2 x_1 + y_2 x_3 - x_2 y_3 + y_3 x_1 - x_3 y_1 \rangle$$

so that:

$$|\vec{u} \times \vec{v}| = | x_2 y_1 - y_2 x_1 + y_2 x_3 - x_2 y_3 + y_3 x_1 - x_3 y_1 |.$$

The area of the triangle is then:

$$\text{Area} = \tfrac{1}{2} | x_2 y_1 - y_2 x_1 + y_2 x_3 - x_2 y_3 + y_3 x_1 - x_3 y_1 |$$

$$= \frac{1}{2} \left| \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \right|$$

$$= | \text{LHS} |. \quad \blacksquare$$