Recitation Handout 4: Proving Vector Identities

The purpose of this recitation is to give you some practice at establishing vector identities (or formulas), especially those that include dot or cross products.

When you attempt your proofs, you may make basic assumptions about the vectors. For example, in a three-dimensional situation you can assume that the vectors you are working with are three-dimensional vectors. However, it is not sufficient to choose a few specific vectors and show that the identity is verified for them.

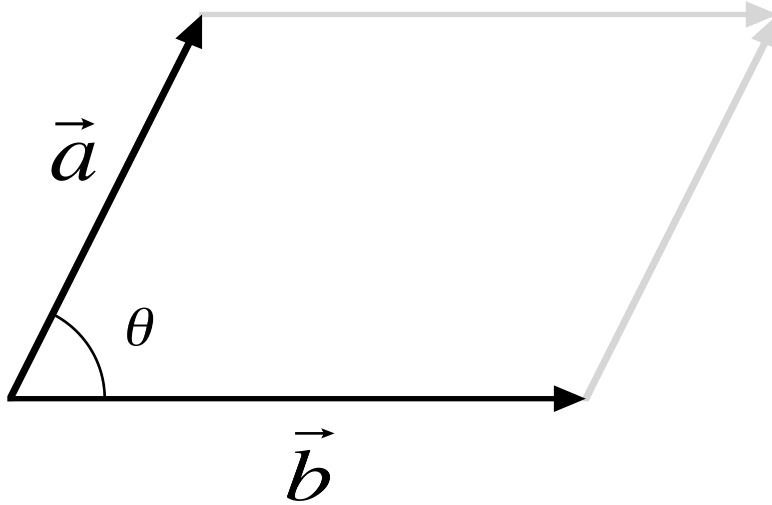
1. Consider two three-dimensional vectors $\vec{a}$ and $\vec{b}$. Show that:

$$\left|\vec{a} \times \vec{b}\right|^2 = \left|\vec{a}\right|^2 \cdot \left|\vec{b}\right|^2 - \left(\vec{a} \bullet \vec{b}\right)^2.$$

2. Consider three three-dimensional vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Show that:

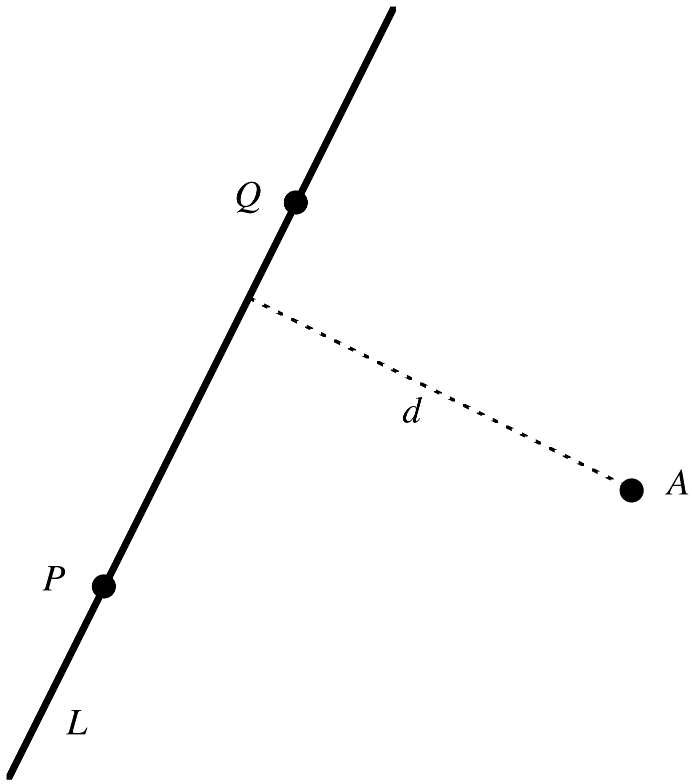
$$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \left(\vec{a}\bullet\vec{c}\right)\vec{b} - \left(\vec{b}\bullet\vec{c}\right)\vec{a}.$$

3. Consider the parallelogram formed by two three-dimensional vectors, $\vec{a}$ and $\vec{b}$, as shown below.



Show that the area of this parallelogram is equal to $\left|\vec{a} \times \vec{b}\right|$.

4. Suppose that P and Q are distinct points on a line, L, in three-dimensional space. Let A be another point in three-dimensional space that does not lies on L. (For example, see the diagram given below.) The points A, P and Q form a triangle. The height of this triangle, d, gives the minimum distance from the line L to the point A.



Let $\vec{u}$ denote the vector from point A to point P, $\vec{v}$ denote the vector from point A to point Q and $\vec{w}$ denote the vector from point P to point Q. Show that:

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{w}|}.$$

5. The determinant of a three-by-three matrix is a number given by the formula:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Consider the triangle with vertices located at $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$. Show that the area of this triangle is equal to one half times the absolute value of the determinant of the matrix:

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}.$$