SOLUTIONS

Recitation Handout 3: Contour Plots and Surfaces in 3D

The purpose of today's activity will be to learn to use a tool (called a contour plot or contour diagram) to visualize the graphs of functions like:

$$z = g(x,y) = x^3 - 3x + y^3 - 3y.$$

The specific learning objectives for this activity are as follows:

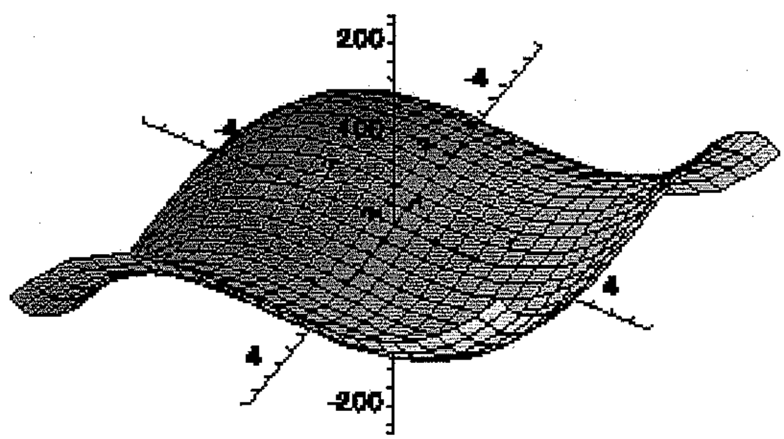
- Learn to read a contour plot and imagine the 3D structures that it represents.
- Create a contour plot from an equation.
- Create a graph in 3D from an equation.

Representing Functions with Several Variables

In this recitation we will concentrate on functions that have two variables (usually x and y) as inputs and one variable (usually z) as an output. For example,

$$z = g(x,y) = x^3 - 3x + y^3 - 3y.$$

The graph of a function like this can be visualized as a surface in three dimensions. If (x, y, z) is a point on this graph, then x and y are the inputs to the function, and the z-coordinate is the output from the function when x and y are plugged in. The graph of $z = g(x, y)$ is shown below.



There are other ways to represent a graph with two inputs and one output. One of these methods involves a two-dimensional representation and is one you are already familiar with from everyday applications such as weather maps (see below).

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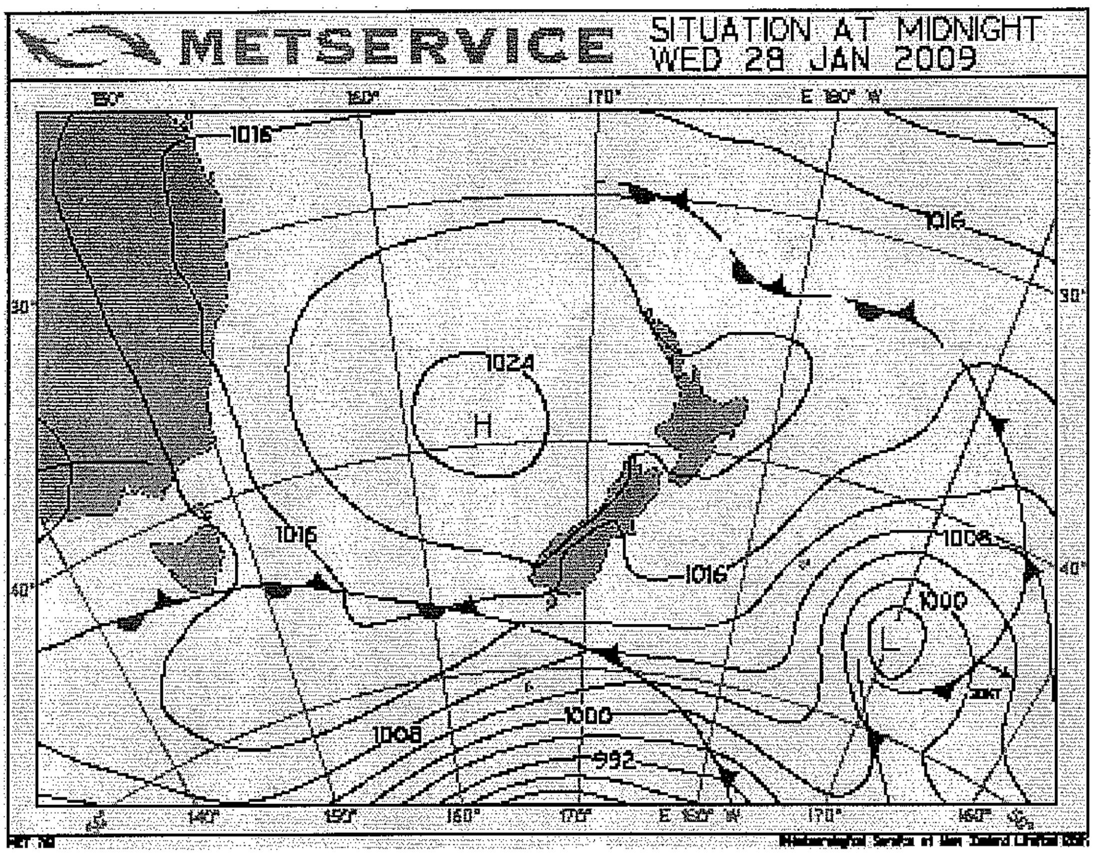

On this diagram, the New Zealand Meteorological Service has graphed a function with two inputs (latitude and longitude) and one output (atmospheric pressure). The way that they have graphed this function is to find all of the input points that give a certain output value (e.g. 1024 kPa) and link these input points with a smooth curve to create a contour.

The same idea can be used to create a two dimensional representation for a mathematical function such as

$$g(x,y) = x^3 - 3x + y^3 - 3y.$$

Exactly how this is done is taken up in the third part of today's recitation.

Reading a Topographic Map

A topographic map is a map of an area that shows only contour lines. The contour lines connect points of the landscape with the same height above sea level. You can look at a topographic map and create a 3D mental image of the landscape. Learning how to do this is the objective of this part of the recitation.

The picture given below shows a topographic map superimposed on a satellite image. As you look at this diagram, try to notice how the pattern of the contours corresponds to terrain features such as tops of hills, valleys, etc.

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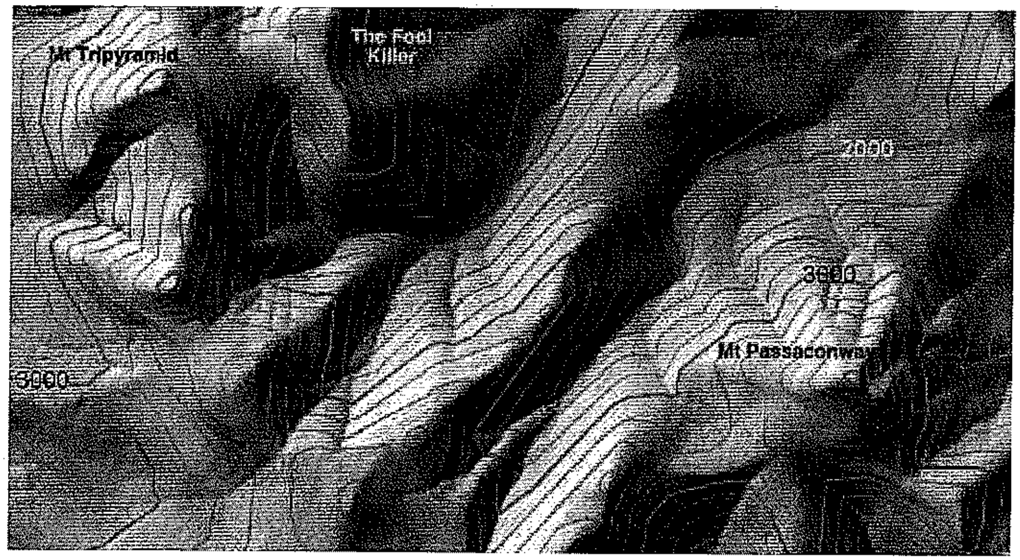

1. Some of the contours have numbers on them (e.g. 2000 or 3000). These numbers are the heights (in feet) above sea level. On the diagram shown above, what is the difference in height between two adjacent contours?

$$\frac{3000 - 2000}{5} = 200 \text{ ft}$$

2. What is the height of Mt. Passaconway? What is the height of Mt. Tripyramid?

Mt. Passaconway $\doteqdot 3000 + 5 \times 200 = 4000$ ft

Mt. Tripyramid $\doteqdot 3000 + 6 \times 200 = 4200$ ft.

3. Assume that in the above diagram, the top of the page represents north. Which is steeper, the northwestern face or the southeastern face of Mt. Passaconway? Briefly explain how you know.

The southeastern face is steeper because the contour lines are closer together.

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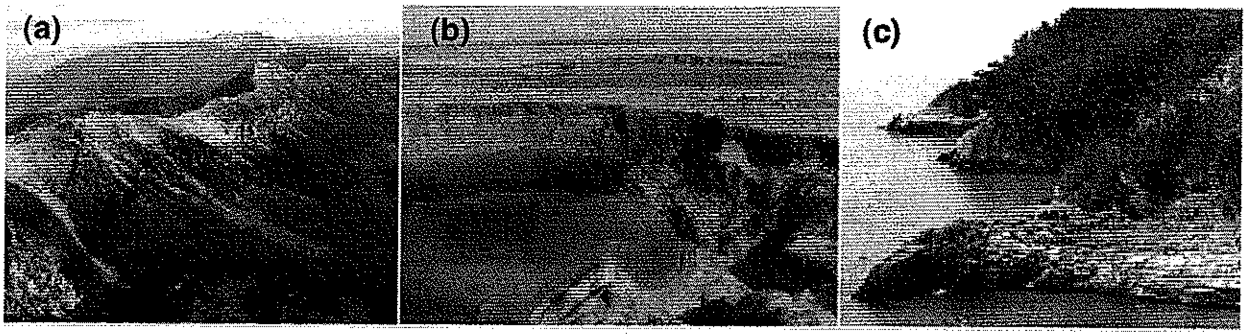
In the next part of the recitation, the objective is to start associating three-dimensional images with the patterns of contours in a contour plot.



Geographical features: (a) Ridge or ridgeline, (b) Sink hole or depression, (c) Spurs.

Geographical features: (d) Saddle (the dip in the middle of the ridge), (e) Hill top, (f) Valley, (g) Cliff face.

4. The diagram given above shows various geographical features and their common names. The topographical map shown below has points of interest labeled with numbers 1-7. By studying the patterns near each number in the topographical map, determine which number is associated with each kind of geographical feature. Record your results in the table provided below.

Number	Corresponding geographical feature
1	Hill top
2	Valley
3	Ridge line
4	Saddle
5	Sink hole.
6	Spur
7	Cliff face

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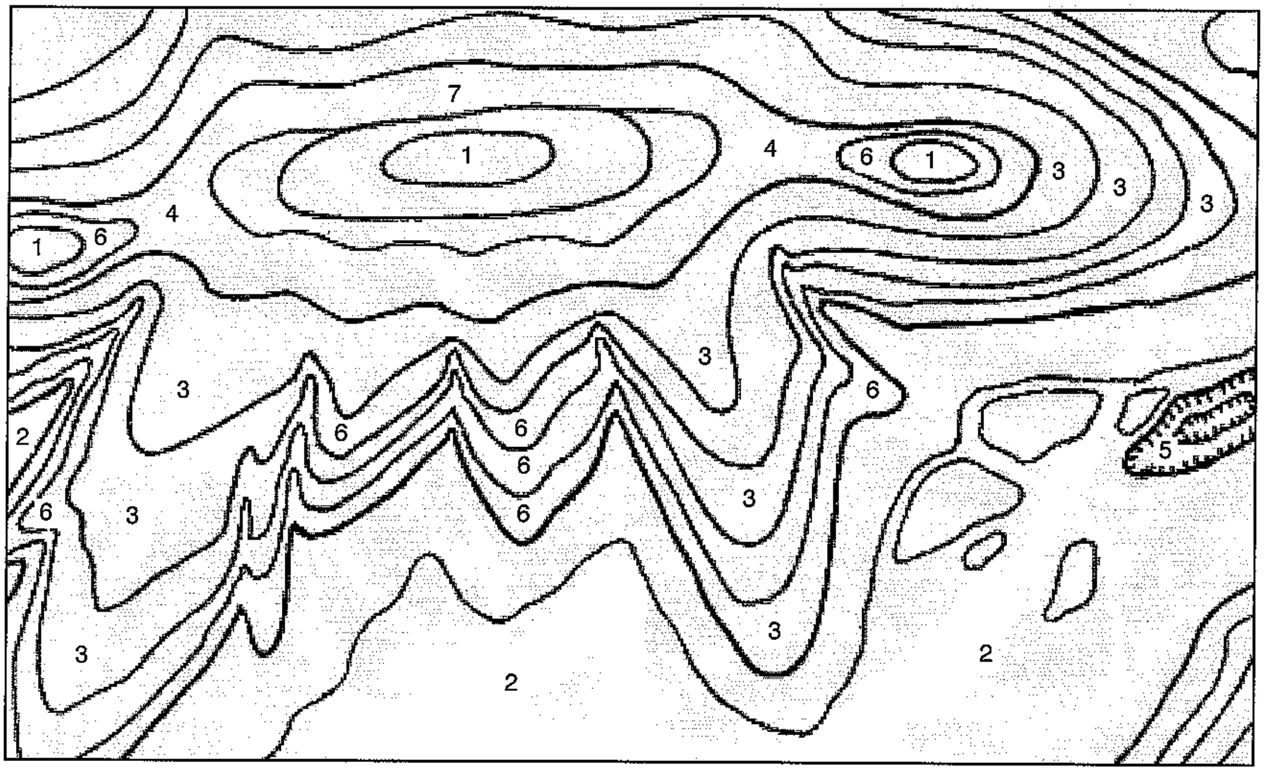

NOTE: On this diagram, contour lines with shading on them (like the contour lines near the number 5) mean that height is lost rather than gained.

Creating a Contour Plot

A contour plot is created from a mathematical function like:

$$g(x,y) = x^3 - 3x + y^3 - 3y$$

by setting $g(x, y)$ equal to a fixed constant, say k. This creates an equation involving only x and y (and possibly constants) that defines a curve in the xy-plane. All of the points (x, y) that lie on this curve will produce an output of k when plugged into $g(x, y)$.

For example, if we take $k = 1$, then adding 4.5 to both sides and completing the square gives:

$$x^3 - 3x + y^3 - 3y = 1$$

$$x^3 - 3x + 1.5^2 + y^3 - 3y + 1.5^2 = 5.5$$

$$(x - 1.5)^2 + (y - 1.5)^2 = 5.5.$$

This is the equation of a circle with center (1.5, 1.5) and radius √5.5. Repeating this procedure for many different values of k gives a contour plot like the one shown below.

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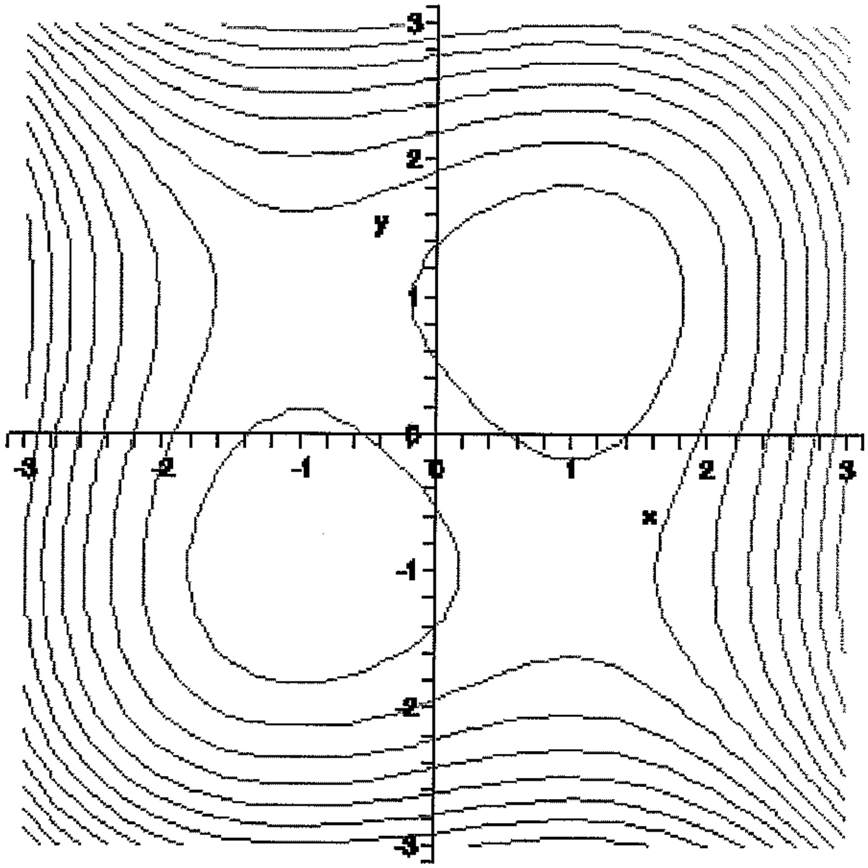


5. The **Cobb-Douglas production function** is a very important function in neoclassical economics. One version of the Cobb-Douglas function can be written as:

$$c(x,y) = \sqrt{x \cdot y}.$$

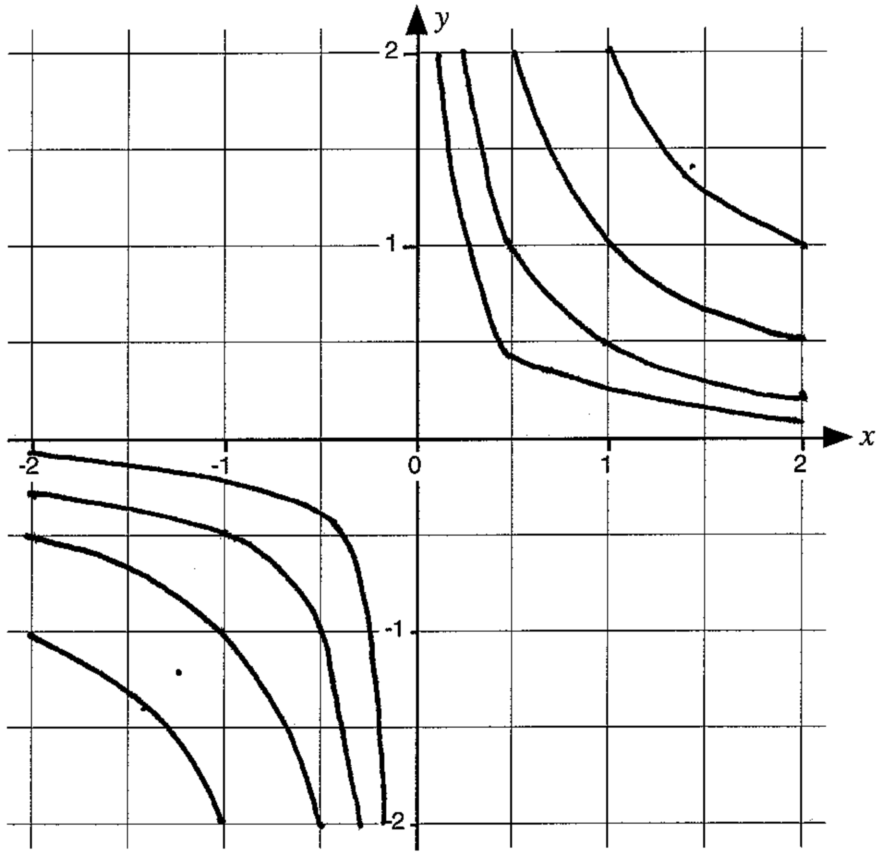
Find the equation for the contours (sometimes called "level curves" for reasons that will soon become clearer) of $c(x, y)$ for $k = 1, 2, 3$.

$k = 1 \quad \sqrt{x \cdot y} = 1 \quad$ so $\quad y = \frac{1}{x}$

$k = 2 \quad \sqrt{x \cdot y} = 2 \quad$ so $\quad y = \frac{2}{x}$

In general, $\quad y = \frac{k}{x}$.

6. Use the axes provided below to sketch a contour plot for the Cobb-Douglas production function.

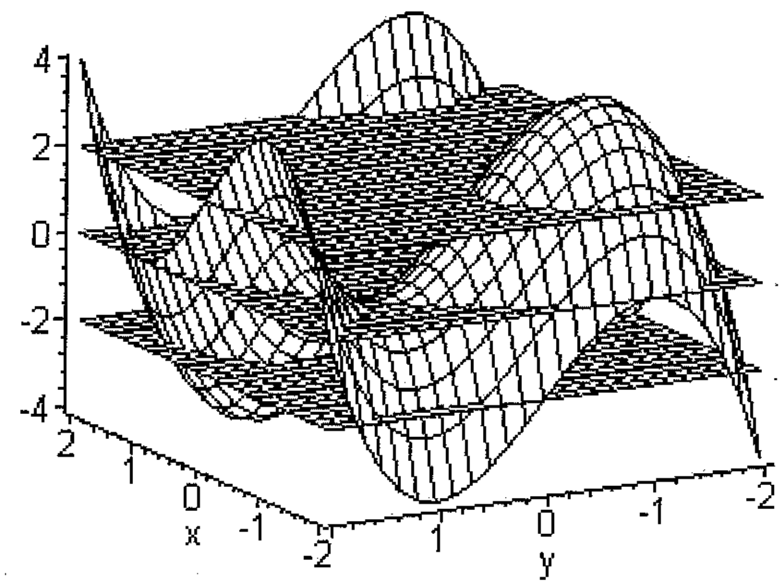


Graphing a Surface in Three Dimensions

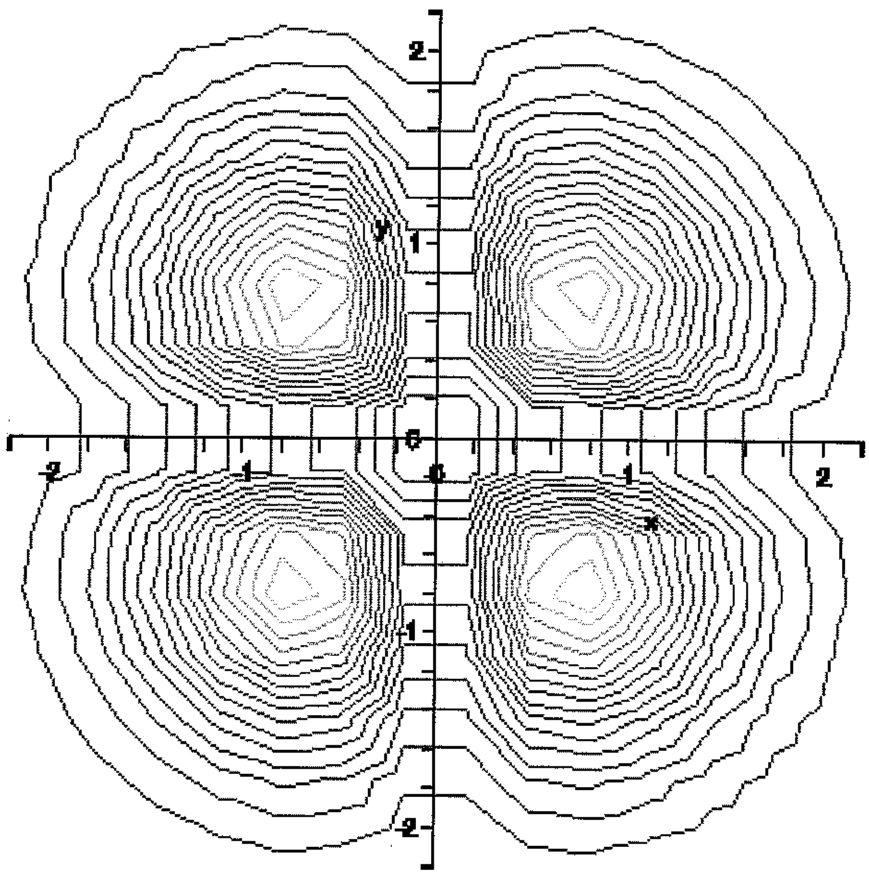
When we have a contour plot for a function $f(x, y)$, it is possible to sketch a 3D graph of the surface $z = f(x, y)$.

To visualize the relationship between a contour plot for a function $f(x, y)$ and the three-dimensional graph $z = f(x, y)$, consider the diagram on the next page.

This diagram shows the graph of $y = f(x, y)$ for some function. Intersecting the graph of the function are horizontal planes with equations of the form $z = k$, for various values of the constant k. The curve formed by the intersection of $z = f(x, y)$ and the plane $z = k$ is the curve in the contour plot defined by the equation $f(x, y) = k$. Just like the lines on a topographic map that show points on the landscape that have the same height, the curves in a contour plot show points on the graph of $z = f(x, y)$ that have the same height. For this reason, the curves in a contour plot are often called **level curves**, because they consist of points that are all on exactly the same level in the graph of $z = f(x, y)$.



7. The contour plot shown below is the contour map for the function $p(x, y)$. You can assume that the graph of $z = p(x, y)$ does not have any sink holes or depressions.



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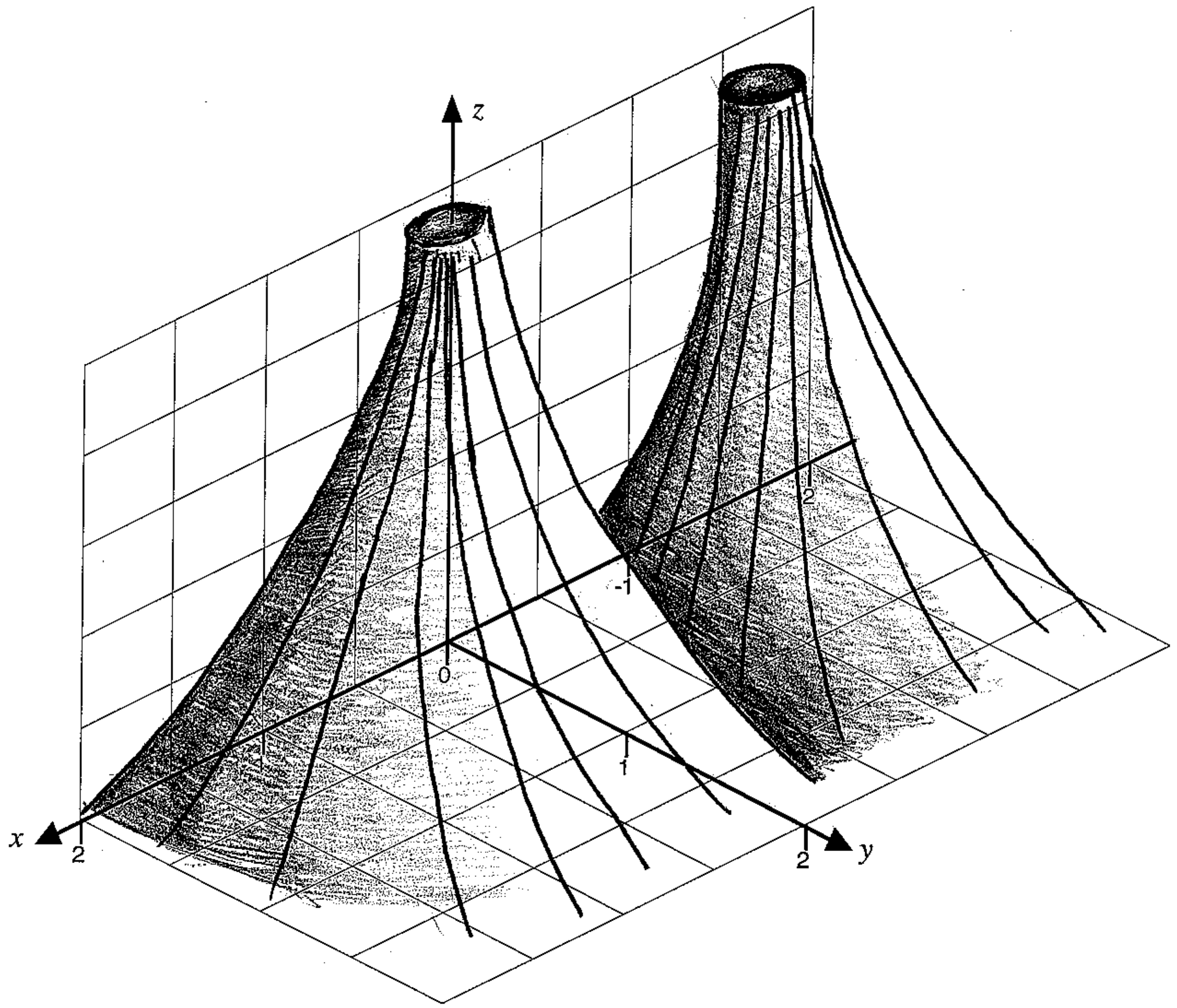
Use the three-dimensional axes provided to sketch a plausible graph for $z = p(x, y)$ for the part of the xy-plane specified by $-2 \le x \le 2$ and $0 \le y \le 2$. (Hint: imagine that you are looking at a topographic map.)



8. Consider the function:

$$q(x,y) = 4 - x^2 - 2y^2.$$

Using several different values of k, find the equations of curves in the contour plot for $q(x, y)$. What kinds of curves are these?

<u>k=1</u> $4 - x^2 - 2y^2 = 1$ so $\dfrac{x^2}{(\sqrt{3})^2} + \dfrac{y^2}{(\sqrt{3/2})^2} = 1$

<u>k=3</u> $4 - x^2 - 2y^2 = 3$ so $\dfrac{x^2}{1^2} + \dfrac{y^2}{(1/\sqrt{2})^2} = 1$

These curves are ellipses centered on (0, 0).

9. Use the axes provided to draw a contour plot for $q(x, y)$ and then use your contour plot to sketch the curve $z = q(x, y)$ for $z \geq 0$.