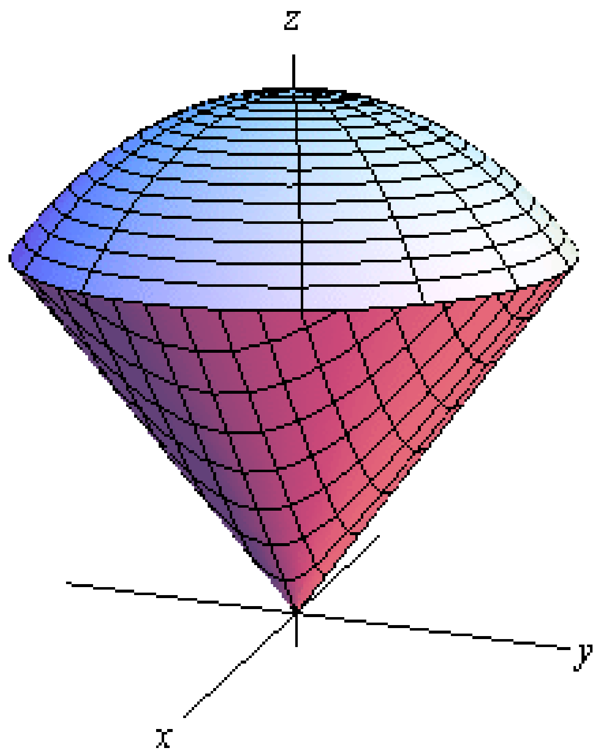
Recitation Handout 11: Visualizing and Setting Up Triple Integrals



Suppose you knew that the density of this object is proportional to z. How would you set up an integral to calculate the mass?

The triple integral is the three-dimensional version of the double integral.

As we have seen in lecture, pictures are very helpful when setting up double integrals, particularly pictures that allow us to visualize the region of integration (usually the shadow cast in the *xy*-plane). Double integrals are comparatively easy for us to set up because we are generally quite good at thinking about two-dimensional geometrical shapes.

For a triple integral, the region of integration is a three-dimensional shape, which is usually not so easy to visualize. (In fact, a lot of our work this semester has concentrated on learning how to visualize three-dimensional shapes, particularly those defined by functions of two or three variables.)

However, when we work to set up the limits of integration for triple integrals we can sometimes reduce the "hard" parts of setting up the limits of integration from three-dimensional to two-dimensional visualization problems. In this recitation, you will practice visualizing the three-dimensional regions of integration for progressively more complicated triple integrals, and learn how to reduce the "hard" parts of these problems to two-dimensional situations.

The main goals in today's recitation is for you to:

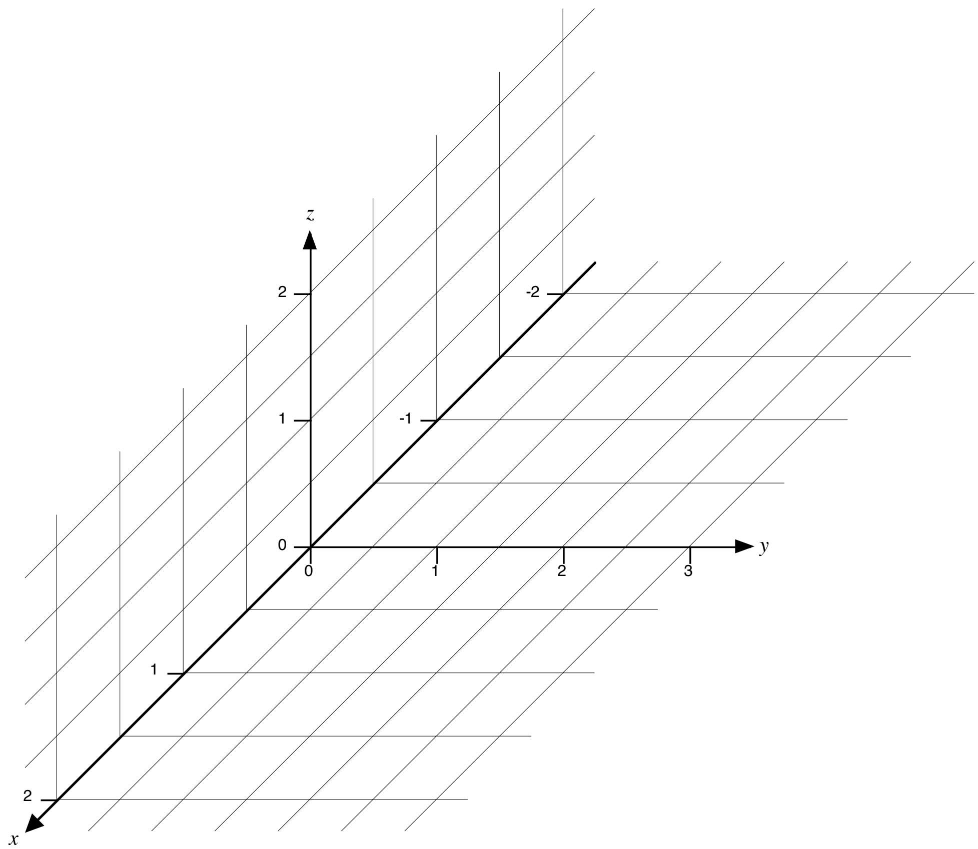
- Practice visualizing shapes in three dimensions,
- Use the three dimensional shapes that you imagine to create limits of integration for triple integrals, and,
- Set up and evaluate triple integrals.

We will begin with three-dimensional regions that are easy to visualize and integrals that are easy to formulate, and then work our way up to integrals that are much more complicated.

Constant Limits of Integration

1. Use the axes provided below to sketch the region of three-dimensional space, A. The region A consists of the points that satisfy all of the following inequalities:

$$-1 \le x \le 1, \qquad 2 \le y \le 3, \qquad \text{and} \qquad 0 \le z \le 1.$$



2. Imagine that the region A, that you drew represents a physical object (with all measurements made in meters), and that the density of this object is given by the function:

$$\delta(x,y,z) = x \cdot y + y \cdot z \ \text{kg/m}^3.$$

Use your picture of A to deduce the limits of integration, and write down a triple integral that will give the **mass** of the object.

Before proceeding to the next question, check your integral with some other people in your class and your TA.

3. Evaluate the triple integral that you set up in Question 2. Include appropriate units with your final answer.

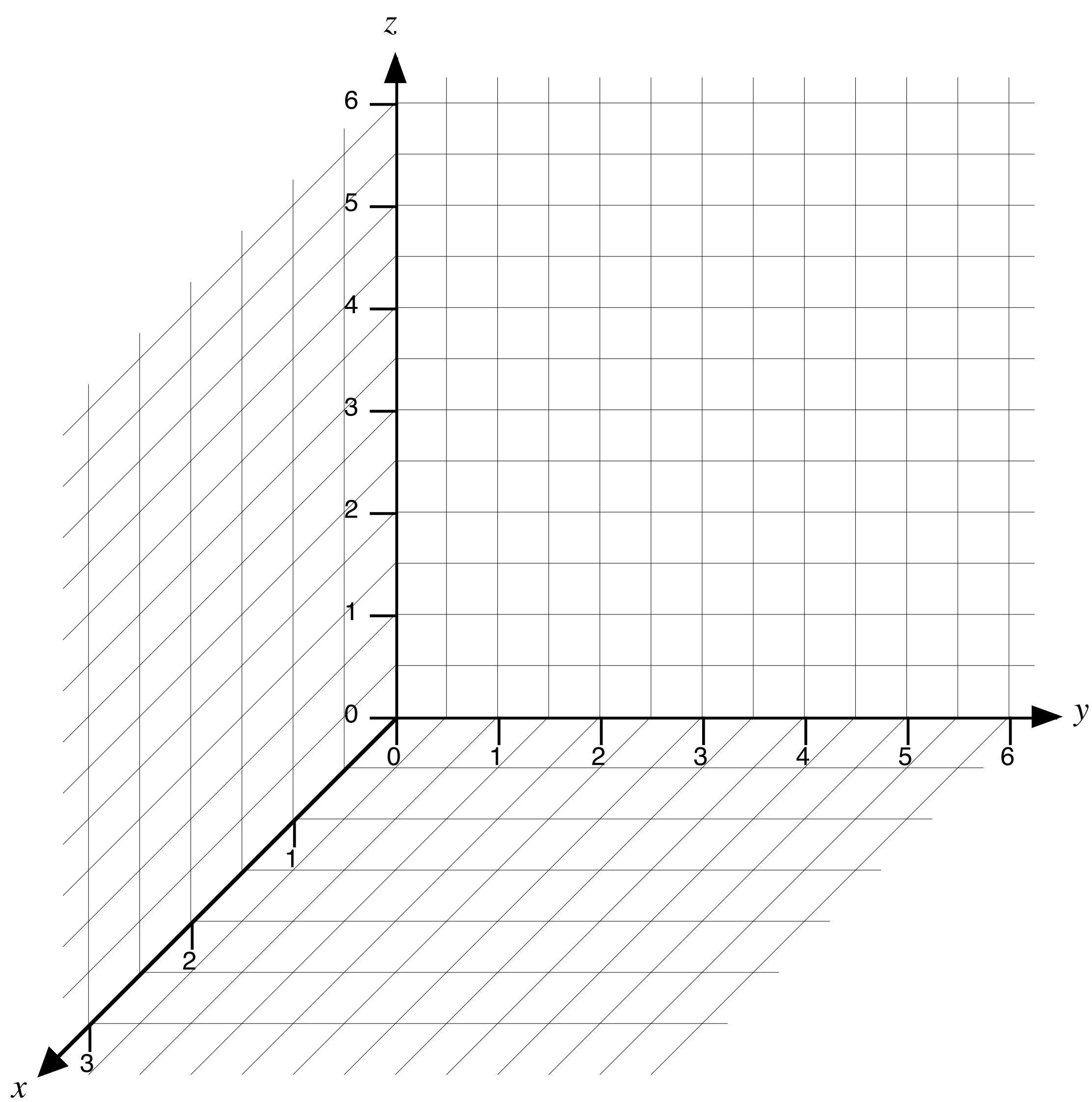
Linear Limits of Integration

4. Next, consider the region B. B is the set of points in three-dimensional space that satisfy all of the following inequalities:

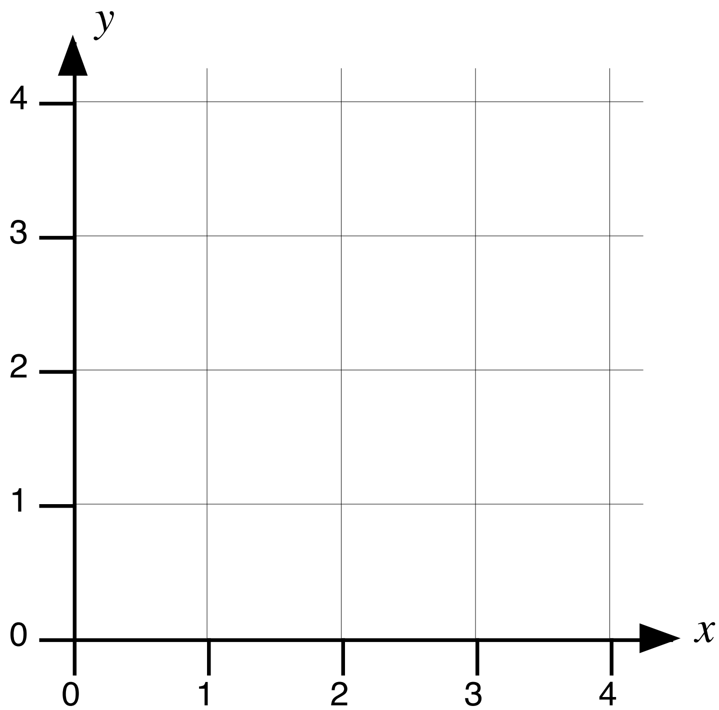
$$x \geq 0 \quad y \geq 0$$

$$0 \leq z \leq 6 - 3x - 2y.$$

Use the set of axes provided below to sketch the region B.



5. Look carefully as the part of the region B that lies in the xy-plane. Use the axes given below to draw this and find equations (involving only x and y) for each of the sides of the shape you draw. This shape will be your region of integration.



6. Imagine that the region B is a physical object (all measurements are in meters) and its density is given by the function:

$$\delta(x, y, z) = z \text{ kg/m}^3.$$

Use your answers to Question 5 and Question 6 to set up a triple integral that will give the **mass** of this object.

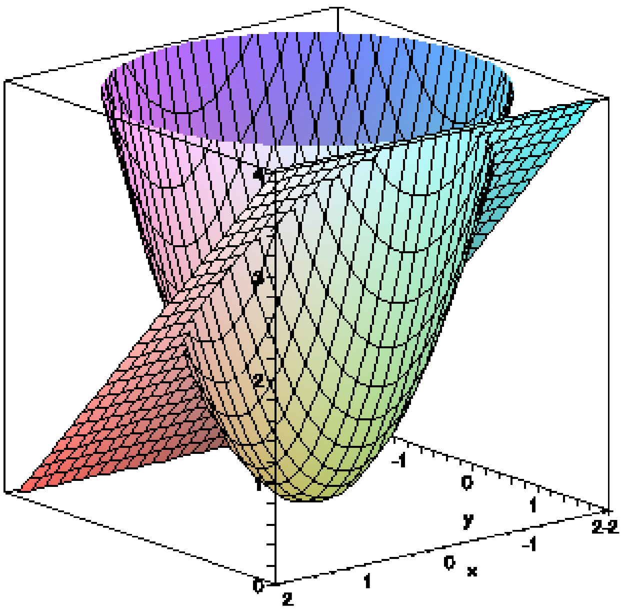
Before proceeding to the next question, check your integral with some other people in your class and your TA. Include appropriate units with your answer.

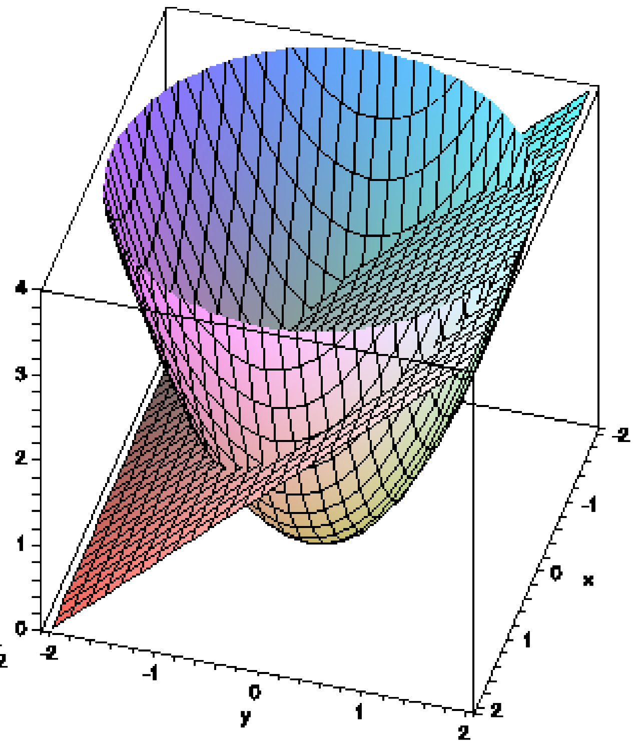
7. Evaluate the triple integral that you set up in Question 6 to find the mass of the object.

Non-Linear Limits of Integration and Changing the Order of Integration

The diagrams given below shows the region C that is bounded by the surfaces:

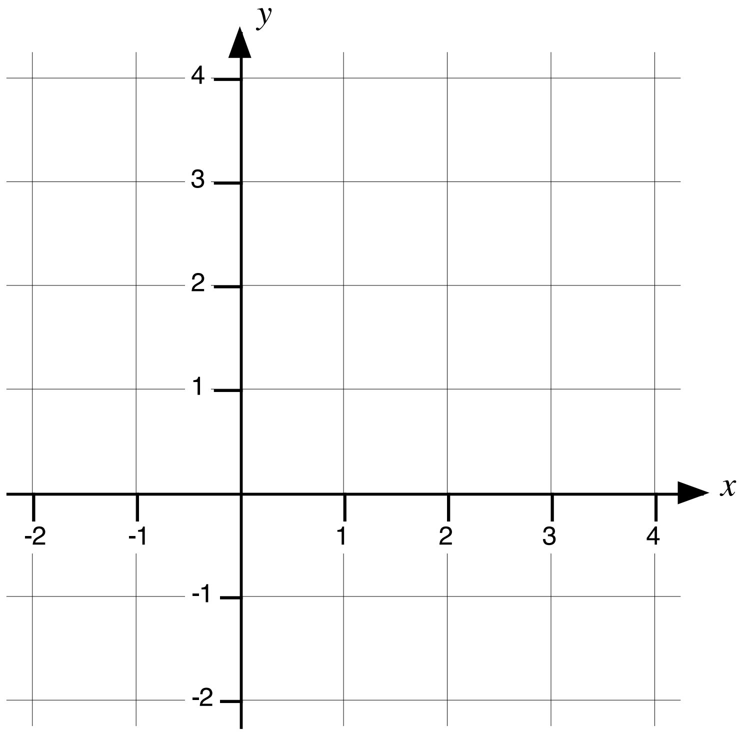
$$z = x^2 + y^2 \qquad \text{and} \qquad z = y + 2.$$





In this final part of the recitation we will be interested in setting up and evaluating a triple integral that gives the volume enclosed by the plane (from above) and the paraboloid (from below).

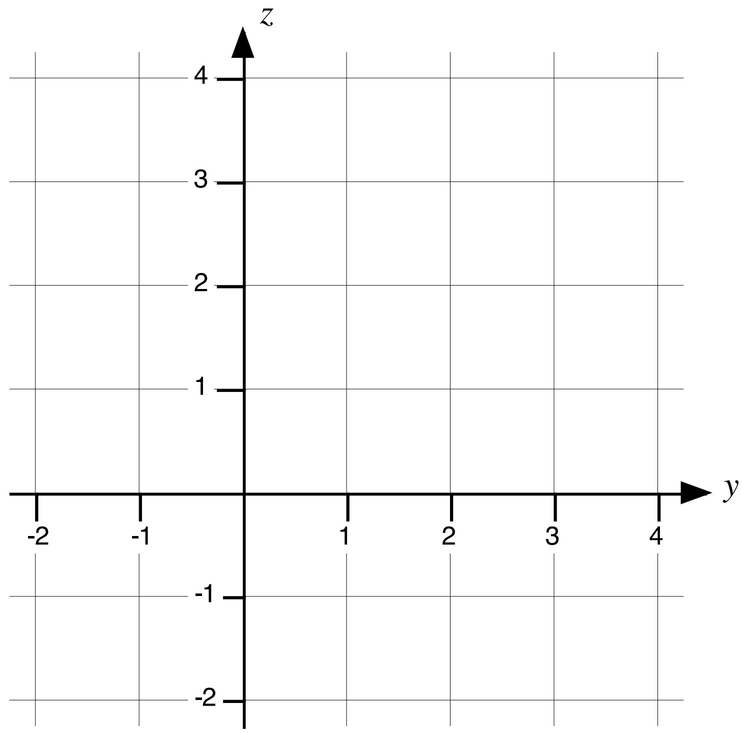
8. By equating these two equations find a formula for the curve in the xy-plane that forms the edge of the region of integration. Use the axes provided on the next page to sketch the region of integration.



9. Set up a triple integral that will give the volume of the region T. (Imagine that you are trying to find the mass of a physical object shaped like T that has a density function $\delta(x, y, z) = 1$. In this case, the integral that you set up for mass will also give you the volume of T.) When you write down your integral, write it so that the order of the differentials at the end is $dz \cdot dy \cdot dx$.

The integral that you wrote down in Question 9 can (at least in theory) be worked out, but it is very awkward to do the calculations. The integral can be made much easier if we depart from out usual practice of using the projection of the three-dimensional shape onto the xy-plane to define the region of integration.

10. Use the axes provided below to draw the projection of the region T onto the yz-plane. Find equations (expressed in terms of y and z only) for each of the curves that form the boundary of your shape.



11. Use your answer to Question 10 to set up a triple integral that will give the volume of the region T. However, when you write down your integral, write it so that the order of the differentials at the end is $dx \cdot dz \cdot dy$.

Before proceeding to the next question, check your integral with some other people in your class and your TA. Include appropriate units with your answer.

12. Evaluate the triple integral that you set up in Question 11 to find the volume of T.

HINT: The most general antiderivative of $(2 + y - y^2)^{3/2}$ is:

$$\frac{-1}{8}(1-2y)\left(2+y-y^2\right)^{3/2} - \frac{27}{64}(1-2y)\sqrt{2+y-y^2} + \frac{243}{128}\sin^{-1}\left(\frac{2y}{3} - \frac{1}{3}\right) + C.$$

(This formula is obtained by completing the square, making a u-substitution and making a trigonometric substitution.)