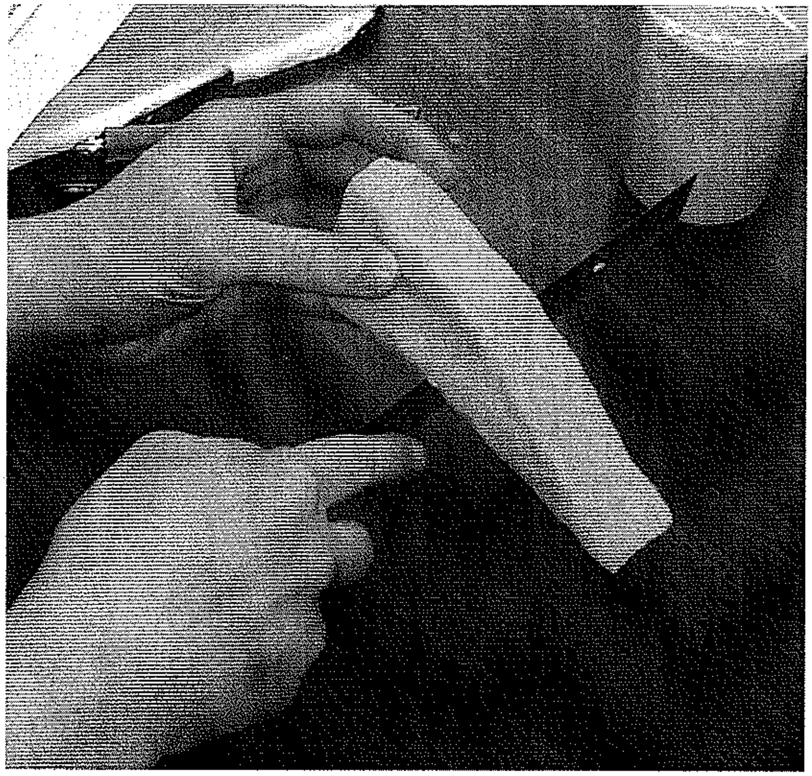
SOLUTIONS

Recitation Handout 10: Applications of Double Integrals

If you completed 21-122 during the Fall semester of 2008 then you probably attended a recitation where you calculated the center of mass (or balance point) of a Play-Doh object like the one shown below.



Today your goal will be calculate the coordinates (x and y) for the center of mass of a much more complicated Play-Doh shape and then verify the accuracy of your calculations by balancing your creation on the tip of a pencil or pen.

Visualizing and building the three dimensional object

In the first part of the recitation, your task will be to work with one or two other students to build the Play-Doh object that you will investigate.

The equipment that you will need to do this is listed below.

- Two tubs of Play-Doh. You will be mixing the Play-Doh so **get two tubs with the same color!**
- A cardboard triangle.
- A plastic ruler.

Your recitation instructor will have all of the equipment that you need.

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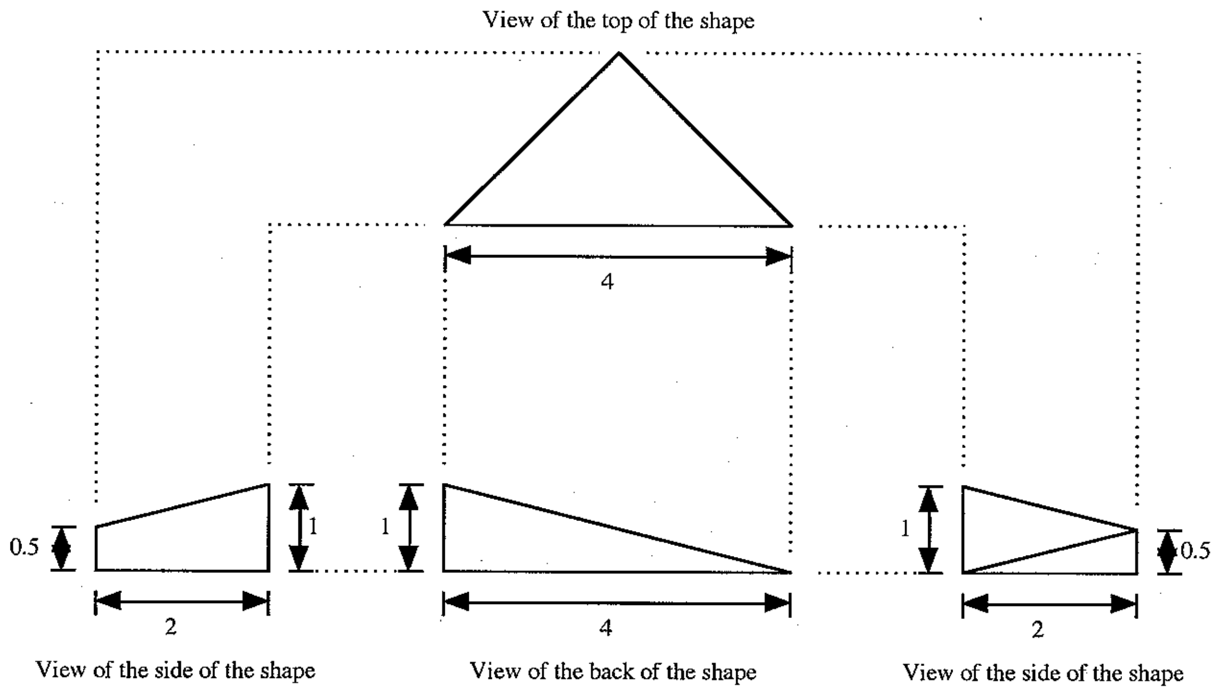
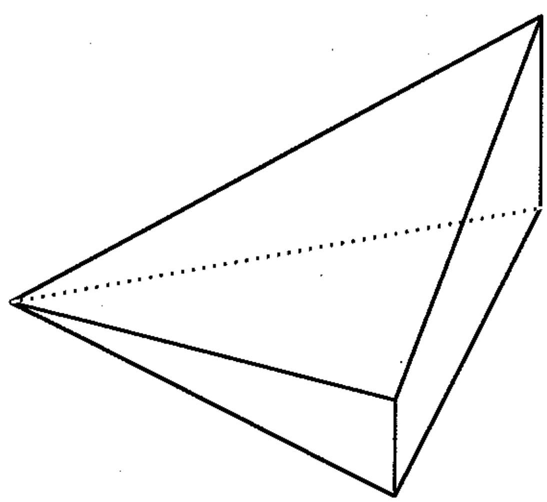
1. Look carefully at the drawings given below. They depict the object that you need to visualize and build. Spend some time studying these pictures and then in collaboration with the other people in your group, build the shape from Play-Doh. Use the cardboard triangle as the base of your shape and the ruler to make sure that all of the dimensions are correct.

The two-dimensional views of the top, back and sides of the shape are shown below. **All dimensions are in units of inches.**



Here is a more three-dimensional view of the front of the shape.



2. When you have finished building your shape, show it to your TA to make sure that you have visualized and constructed the shape correctly.

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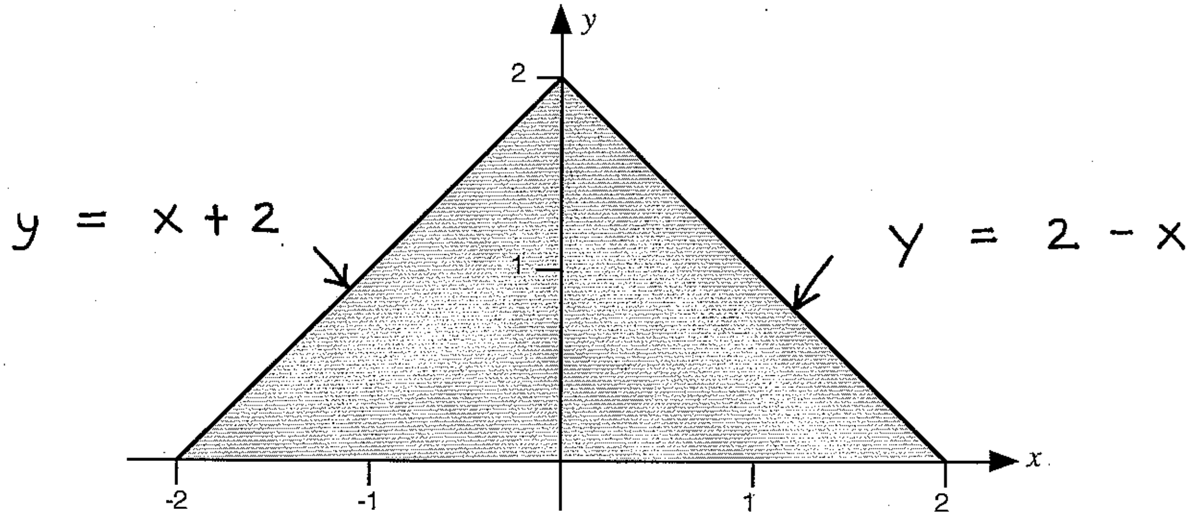
Finding the mass of the Play-Doh shape

The Play-Doh shape that you have created simulates a two-dimensional triangular plate



with density $\delta(x, y)$ that varies from point to point. If you have built your shape correctly then the density function that your shape simulates is:

$$\delta(x,y) = \frac{-kx}{4} + \frac{k}{2},$$

where $k > 0$ is a positive constant.

3. Write down a pair of integrals that (when evaluated) will give the total mass of the triangular plate shown above. (Note that as $\delta(x, y)$ really only depends on x this can be a pair of dx, $dxdy$ or $dydx$ integrals – your choice!)

$$\text{Mass} = \int_{-2}^{0}\int_{0}^{x+2}\left(\frac{-kx}{4}+\frac{k}{2}\right)dy\,dx + \int_{0}^{2}\int_{0}^{2-x}\left(\frac{-kx}{4}+\frac{k}{2}\right)dy\,dx$$

$$= \int_{0}^{2}\int_{y-2}^{2-y}\left(\frac{-kx}{4}+\frac{k}{2}\right)dx\,dy$$

$$= \int_{-2}^{0}\left(\frac{-kx}{4}+\frac{k}{2}\right)(x+2)\,dx + \int_{0}^{2}\left(\frac{-kx}{4}+\frac{k}{2}\right)(2-x)\,dx$$

Before proceeding to the next question, check your integrals with some other people in your class and your TA.

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4. Evaluate your integrals to find an expression for the total mass of the triangular plate. Your answer should include the constant k.

$$\text{Mass} = \int_{-2}^{0}\int_{0}^{x+2}\left(\frac{-kx}{4}+\frac{k}{2}\right)dy\,dx + \int_{0}^{2}\int_{0}^{2-x}\left(\frac{-kx}{4}+\frac{k}{2}\right)dy\,dx$$

$$= \int_{-2}^{0}\left[\left(-\frac{kx}{4}+\frac{k}{2}\right)y\right]_{0}^{x+2}dx + \int_{0}^{2}\left[\left(\frac{-kx}{4}+\frac{k}{2}\right)\cdot y\right]_{0}^{2-x}dx$$

$$= \int_{-2}^{0}\frac{k}{4}(2-x)(2+x)\,dx + \int_{0}^{2}\frac{k}{4}(2-x)(2-x)\,dx$$

$$= \int_{-2}^{0}\frac{k}{4}(4-x^2)\,dx + \int_{0}^{2}\frac{k}{4}(4-4x+x^2)\,dx$$

$$= \frac{k}{4}\left[4x-\frac{1}{3}x^3\right]_{-2}^{0} + \frac{k}{4}\left[4x-2x^2+\frac{1}{3}x^3\right]_{0}^{2}$$

$$= 2k$$

5. Take your Play-Doh shape to the front of the room and find its mass. (Your recitation instructor will have a set of scales that you can use to find the mass.)

The one I built weighed exactly 50 g.

6. By comparing the expression that you calculated in Question 4 to the measurement that you made in Question 5, determine the value of the constant k that appears in the formula for $\delta(x, y)$.

$2k = 50\text{ g}$ so $k = 25.$

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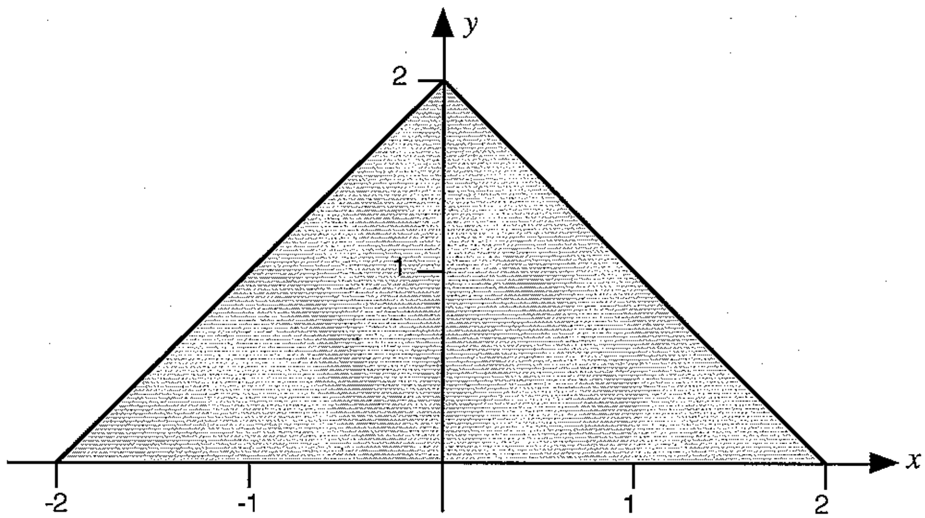
Calculating the *x*-coordinate of the center of mass

The center of mass of an object is its balance point. If we attempt to balance your Play-Doh shape by putting the cardboard base on the tip of a pencil, the balance point that we are looking for will have an x and a y coordinate. In this (and the next) part of the recitation, you will calculate these coordinates.

If we use the symbol T to represent the triangular plate:



Then the x-coordinate of the center of mass (usually written $\bar{x}$) is given by the formula:

$$\bar{x} = \frac{\iint_T x \cdot \delta(x,y) \cdot dx \cdot dy}{\text{Total mass}}.$$

7. Set up two integrals that will give the value of $\iint_T x \cdot \delta(x,y) \cdot dx \cdot dy$ when added together.

   We recommend that you set the integrals up as *dydx* integrals but it is also possible to set them up as *dx* or *dxdy* integrals.

$$\iint_T x \cdot \delta(x,y)\, dx\, dy = \int_0^2 \int_{y-2}^{0} \left(\frac{-25x^2}{4} + \frac{25x}{2} \right) dx\, dy$$

$$+ \int_0^2 \int_0^{2-y} \left(-\frac{25x^2}{4} + \frac{25x}{2} \right) dx\, dy$$

Before proceeding to the next question, check your integrals with some other people in your class and your TA.

8. Evaluate the two integrals that you set up and add them together to calculate:

$$\iint_T x \cdot \delta(x,y) \cdot dx \cdot dy.$$

$$\iint_T x \cdot \delta(x,y)\, dx\, dy = \int_0^2 \left[\frac{-25}{12} x^3 + \frac{25}{4} x^2 \right]_{y-2}^{0} dy$$

$$+ \int_0^2 \left[\frac{-25}{12} x^3 + \frac{25}{4} x^2 \right]_0^{2-y} dy$$

$$= \int_0^2 \frac{25}{12} (y-2)^3 - \frac{25}{4} (y-2)^2 \, dy$$

$$+ \int_0^2 \frac{-25}{12} (2-y)^3 + \frac{25}{4} (2-y)^2 \, dy$$

$$= \left[\frac{25}{48} (y-2)^4 - \frac{25}{12} (y-2)^3 \right]_0^2$$

$$+ \left[\frac{25}{48} (2-y)^4 - \frac{25}{12} (2-y)^3 \right]_0^2$$

$$= -16\,^2/_3.$$

9. Use the total mass that you measured in Question 5 to calculate $\bar{x}$.

$$\bar{x} = \frac{-16\,^2/_3}{50} = -\,^1/_3.$$

Calculating the *y*-coordinate of the center of mass

Then the *x*-coordinate of the center of mass (usually written $\bar{y}$) is given by the formula:

$$\bar{y} = \frac{\iint_T y \cdot \delta(x,y) \cdot dx \cdot dy}{\text{Total mass}}.$$

10. Set up two integrals that will give the value of $\iint_T y \cdot \delta(x,y) \cdot dx \cdot dy$ when added together.

We recommend that you set the integrals up as *dydx* integrals but it is also possible to set them up as *dx* or *dxdy* integrals.

$$\iint_T y \cdot \delta(x,y)\, dy\, dx = \int_{-2}^{0} \int_{0}^{x+2} \left(\frac{-25x}{4} + \frac{25}{2} \right) y \, dy\, dx$$

$$+ \int_{0}^{2} \int_{0}^{2-x} \left(\frac{-25x}{4} + \frac{25}{2} \right) y \, dy\, dx$$

Before proceeding to the next question, check your integrals with some other people in your class and your TA.

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11. Evaluate the two integrals that you set up and add them together to calculate:

$$\iint_T y \cdot \delta(x,y) \cdot dx \cdot dy.$$

$$\iint_T y \cdot \delta(x,y)\, dy\, dx = \int_{-2}^{0} \left[\left(\frac{-25x}{4} + \frac{25}{2}\right)\frac{y^2}{2}\right]_0^{x+2} dx$$

$$+ \int_{0}^{2} \left[\left(\frac{-25x}{4} + \frac{25}{2}\right)\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \frac{25}{8}\int_{-2}^{0} (2-x)(x+2)^2\, dx$$

$$+ \frac{25}{8}\int_{0}^{2} (2-x)^3\, dx$$

$$= \frac{25}{8}\left[-\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{4}{2}x^2 + 8x\right]_{-2}^{0}$$

$$+ \frac{25}{8}\left[-\frac{1}{4}(2-x)^4\right]_0^2$$

$$= 33\tfrac{1}{3}$$

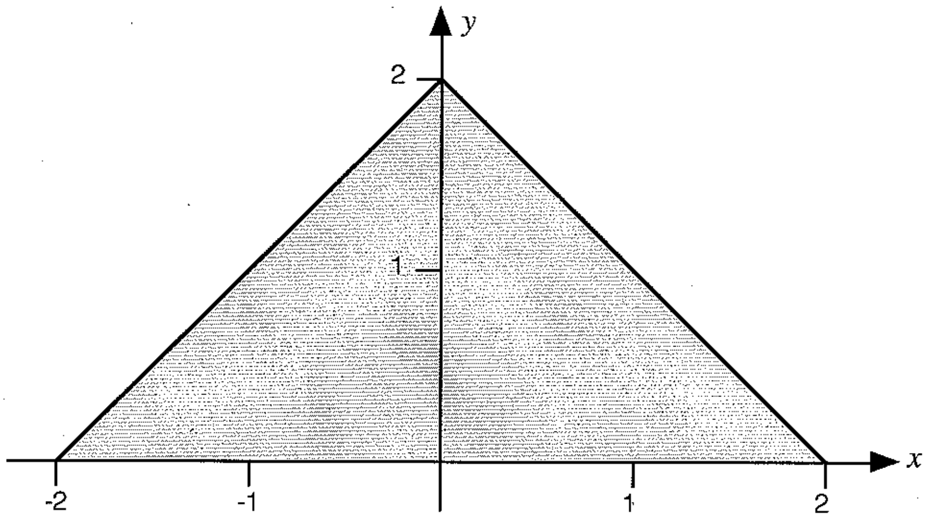
12. Use the total mass that you measured in Question 5 to calculate $\bar{y}$.

$$\bar{y} = \frac{33\tfrac{1}{3}}{50} = \frac{2}{3}$$

Checking your prediction

Now you get to see whether your hard work in setting up and evaluating the integrals has yielded correct answers.

13. Using the *xy*-axes that appear in the picture:



as a reference, find the point on the cardboard base of your Play-Doh shape that has the x and y coordinates $(\bar{x}, \bar{y})$. This should be the balance point (or close to it) of your shape.

$$(\bar{x}, \bar{y}) = \left(-1/3,\ 2/3\right)$$

both measured in inches

14. Find a sharp, pointy object (such as the tip of a pencil or pen; your finger is probably a bit too soft and large but you can try it). Try to balance your Play-Doh shape using the center of mass $(\bar{x}, \bar{y})$ that you found. If you have to adjust the balance point very much to get the Play-Doh shape to balance, try to explain why.

The density formula we used does not take the mass of the cardboard triangle into account.