# MATH 259 – THIRD UNIT TEST

**Friday, April 24, 2009.**

NAME: SOLUTIONS

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|---|---|---|---|---|
| A | C | D | G | H |
| B | F | E | | |

**Instructions:**

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

| Problem | Total | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 16 | |
| 3 | 24 | |
| 4 | 10 | |
| 5 | 20 | |
| 6 | 15 | |
| Total | 100 | |

SOLUTIONS



1. **15 Points. CLEARLY INDICATE YOUR ANSWERS.**

In this problem you will be concerned with the solid $S$ bounded by:

- The elliptic paraboloid $x^2 + 2y^2 + z = 16$,
- The plane $x = 2$,
- The plane $y = 2$, and,
- The three coordinate planes.

**NOTE:** You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

**(a)** **(7 points)** Write down a **triple integral** that will give the volume of the solid $S$. You do not have to evaluate the integral in Part (a).

There are several ways to write down the answer, depending on how you order dx, dy and dz.

$$\text{Volume} = \int_0^2 \int_0^2 \int_0^{16-x^2-2y^2} 1 \, dz\, dy\, dx.$$

**(b)** **(8 points)** Find the volume of the solid $S$.

$$\text{Volume} = \int_0^2 \int_0^2 \int_0^{16-x^2-2y^2} 1 \, dz\, dy\, dx$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dy\, dx$$

$$= \int_0^2 \left[16y - yx^2 - \frac{2}{3}y^3\right]_0^2 dx$$

$$= \int_0^2 \left(\frac{80}{3} - 2x^2\right) dx$$

$$= \left[\frac{80}{3}x - \frac{2}{3}x^3\right]_0^2$$

$$= 48.$$

**2. 16 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.**

Use Lagrange Multipliers to find the maximum and minimum values of the function:

$$f(x,y) = x^2 \cdot y$$

subject to the constraint:

$$x^2 + 2y^2 = 6.$$

Show your work and record your answers in the space provided below.

The constraint function is $g(x,y) = x^2 + 2y^2 - 6 = 0$.

$\nabla f = \langle 2xy,\ x^2 \rangle \qquad \nabla g = \langle 2x,\ 4y \rangle.$

The system of equations is: $\nabla f = \lambda \nabla g$, or:

$$2xy = 2\lambda x$$
$$x^2 = 4\lambda y.$$

Case 1: $x = 0$ $\qquad y = \pm\sqrt{3}$

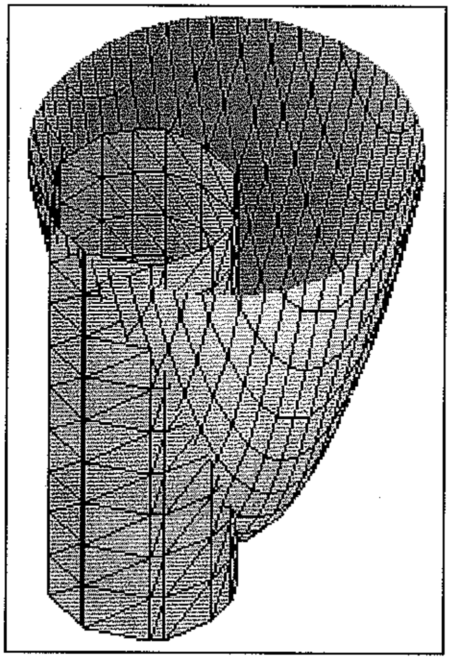
Case 2: $x \neq 0$ $\qquad y = \lambda$ and $x^2 = 4\lambda^2$ so that $x = \pm 2y$. Plugging this into the constraint gives $y = \pm 1$.

| $x$ | $y$ | $f(x,y)$ | Comments |
|---|---|---|---|
| 0 | $\sqrt{3}$ | 0 | |
| 0 | $-\sqrt{3}$ | 0 | |
| 2 | 1 | 4 | Global maximum |
| −2 | 1 | 4 | |
| −2 | −1 | −4 | Global minimum |
| 2 | −1 | −4 | |

**MAXIMUM VALUE OF $f(x, y)$:** 4

**MINIMUM VALUE OF $f(x, y)$:** −4

**3. 24 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**



The diagram given on the left shows part of the paraboloid:

$$z = x^2 + y^2$$

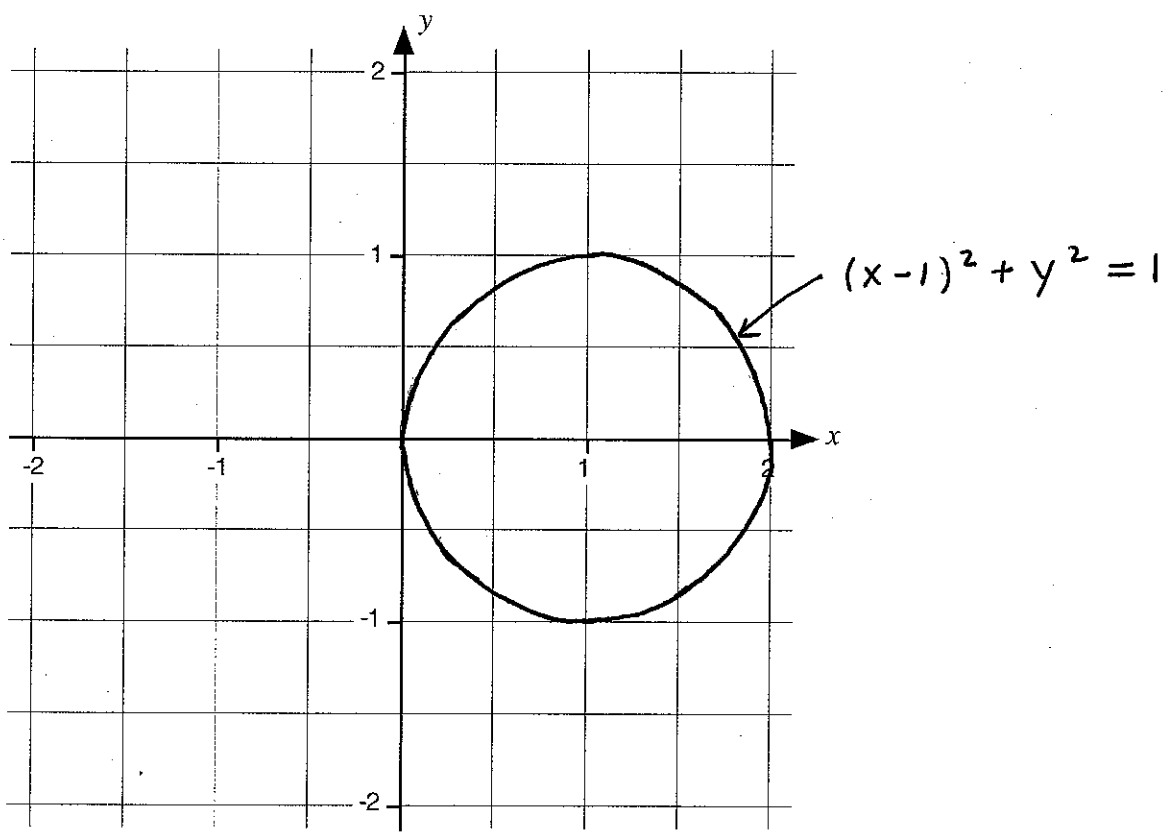
and part of the cylinder:

$$(x-1)^2 + y^2 = 1.$$

In this problem you will be concerned with the solid $D$ that is inside the cylinder, lies under the paraboloid and above the $xy$-plane.

**NOTE:** You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

**(a)** **(4 points)** Use the axes provided below to draw the "shadow" cast on the $xy$-plane by the solid $D$.

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**SHOW YOUR WORK. NO WORK = NO CREDIT.**

The diagram on the previous page shows part of the paraboloid $z = x^2 + y^2$ and part of the cylinder $(x-1)^2 + y^2 = 1$.

**(b)** **(5 points)** Write down a **double integral** expressed in Cartesian coordinates that will give the **volume** of the solid $D$. You do not have to evaluate this integral.

$$\text{Volume} = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} (x^2+y^2)\, dy\, dx \quad \text{, or,}$$

$$\text{Volume} = \int_{-1}^{1} \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1+y^2}} (x^2+y^2)\, dx\, dy.$$

**(c)** **(5 points)** Convert the equation of the circle in the $xy$-plane:

$$(x-1)^2 + y^2 = 1$$

from Cartesian to polar coordinates. Clearly indicate your final answer.

$$x^2 - 2x + 1 + y^2 = 1$$

$$x^2 + y^2 = 2x$$

$$r^2 = 2r\cos(\theta)$$

$$r = 2\cos(\theta).$$

In polar coordinates, the equation of the circle is:

$$r = 2 \cdot \cos(\theta).$$

*Continued on the next page.*

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**SHOW YOUR WORK. NO WORK = NO CREDIT.**

You may use the following integral formula without having to verify it:

$$\int \cos^5(x)dx = \tfrac{1}{5}\cos^4(x)\sin(x) + \tfrac{4}{15}\cos^2(x)\sin(x) + \tfrac{8}{15}\sin(x) + C.$$

You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

**(c)** **(4 points)** Suppose that the solid $D$ is made from a material with density: $\delta(x,y) = \sqrt{x^2+y^2}$. Part of an integral that will give the **mass** of the solid is shown below. Complete this expression by adding limits of integration in the spaces shown. Note that 0 and $2\pi$ are **not** limits of integration for $d\theta$.

$$\int_{\boxed{-\frac{\pi}{2}}}^{\boxed{\frac{\pi}{2}}} \int_{\boxed{0}}^{\boxed{2\cdot\cos(\theta)}} \int_0^{r^2} \delta(x,y)\cdot r\cdot dz\cdot dr\cdot d\theta$$

**(d)** **(6 points)** Calculate the mass of the solid $D$. Clearly indicate your final answer. You do not need to include units in your final answer.

$$\text{Mass} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} \int_0^{r^2} r^2 \, dz \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^4 \, dr \, d\theta$$

$$= \frac{32}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5(\theta) \, d\theta$$

$$= \frac{32}{5}\left[\frac{1}{5}\cos^4(\theta)\sin(\theta) + \frac{4}{15}\cos^2(\theta)\sin(\theta) + \frac{8}{15}\sin(\theta)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{512}{75} \approx 6.8266$$

4. **10 Points. SHOW YOUR WORK. CLEARLY INDICATE YOUR ANSWERS.**

In this problem, the path $C$ is part of the graph of the parabola $x = 4 - y^2$ from the point (–5, –3) to the point (0, 2). Calculate the exact value of the path integral:

$$\int_C y^2 \cdot dx + x \cdot dy$$

Clearly indicate your final answer.

**NOTE:** You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

Parametric equations:

$$x(t) = 4 - t^2 \qquad -3 \leq t \leq 2.$$
$$y(t) = t$$

$$\int_C y^2\,dx = \int_{-3}^{2} t^2(-2t)\,dt = \left[\frac{-1}{2}t^4\right]_{-3}^{2}$$

$$= \frac{65}{2}$$

$$\int_C x\,dy = \int_{-3}^{2} (4-t^2)(1)\,dt = \left[4t - \frac{1}{3}t^3\right]_{-3}^{2}$$

$$= \frac{25}{3}$$

$$\int_C y^2\,dx + x\,dy = \frac{65}{2} + \frac{25}{3} = \frac{245}{6}.$$

SOLUTIONS



5. **20 Points. CLEARLY INDICATE YOUR FINAL ANSWERS.**

(a) **(5 points)** A surface is defined by the following equation in spherical coordinates:

$$\rho = \cos(\varphi).$$

Convert this equation to Cartesian coordinates and identify the surface (e.g. plane, elliptic paraboloid, etc.).

$$\rho^2 = \rho\cos(\varphi)$$

$$x^2 + y^2 + z^2 = z$$

$$x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$$

$$x^2 + y^2 + (z - 1/2)^2 = 1/4.$$

This surface is a sphere.

(b) **(7 points)** Consider the cone defined by the equation:

$$z = \sqrt{x^2 + y^2}.$$

Convert this equation to spherical coordinates.

$$\rho\cos(\varphi) = \sqrt{\rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta)}$$

$$\rho\cos(\varphi) = \sqrt{\rho^2\sin^2(\varphi)}$$

$$\rho\cos(\varphi) = \rho\sin(\varphi)$$

$$\tan(\varphi) = 1$$

$$\varphi = \pi/4.$$

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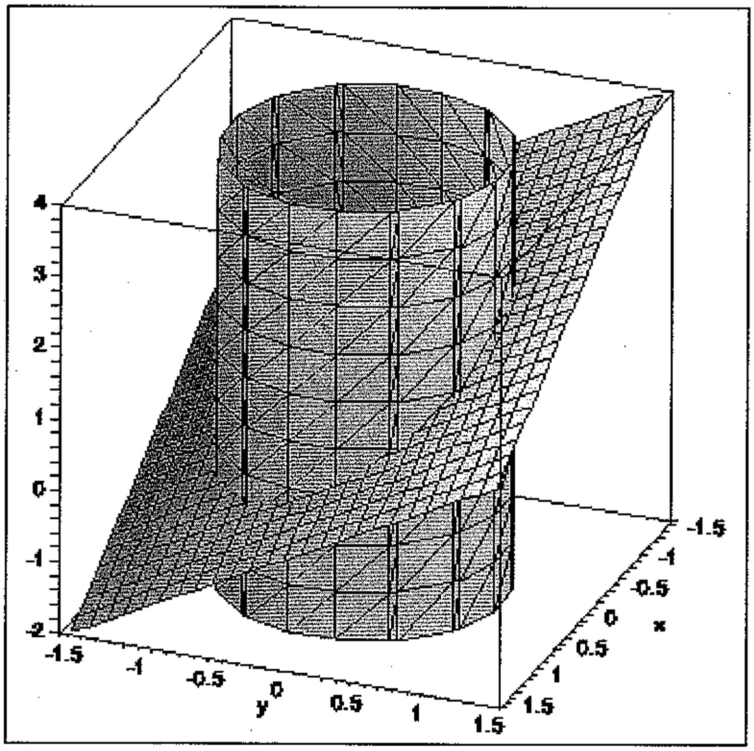
**CLEARLY INDICATE YOUR FINAL ANSWERS.**

(c) **(8 points)** Consider the solid $T$ bounded below by the surface $\rho = \cos(\varphi)$ and above by the surface $z = \sqrt{x^2 + y^2}$. Write down a **triple integral in spherical coordinates** that will give the volume of the solid $T$. You do not need to evaluate this integral.

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos(\varphi)} 1 \, \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta$$

(There are many other correct ways to write down this volume.)

**6. 15 Points. SHOW YOUR WORK.**




The curve of intersection between the cylinder:

$$x^2 + y^2 = 1$$

and the plane:

$$x - y + z = 1$$

is shown in the diagram on the left. Find the maximum and minimum values of the function:

$$f(x,y,z) = x + 2y + 3z$$

on the curve of intersection shown below.

Record your final answers in the space provided on the next page.

We will use Lagrange Multipliers to find the maximum and minimum values of $f(x, y, z)$ subject to the constraints:

$$g(x, y, z) = x^2 + y^2 - 1 = 0$$

$$h(x, y, z) = x - y + z - 1 = 0.$$

$$\left.\begin{aligned} \nabla f &= \langle 1, 2, 3 \rangle \\ \nabla g &= \langle 2x, 2y, 0 \rangle \\ \nabla h &= \langle 1, -1, 1 \rangle \end{aligned}\right\}$$

So the system of equations $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$ is:

$$1 = 2\lambda_1 x + \lambda_2$$

$$2 = 2\lambda_1 y - \lambda_2$$

$$3 = \lambda_2$$

***Additional space for your work on the next page.***

**SHOW YOUR WORK.**

The curve of intersection between the cylinder:

$$x^2 + y^2 = 1$$

and the plane:

$$x - y + z = 1$$

is shown on the previous page. Find the maximum and minimum values of the function:

$$f(x,y,z) = x + 2y + 3z$$

on the curve of intersection. Record your final answers in the space provided below.

As $\lambda_2 = 3$ we can re-write the remaining equations as:

$$\lambda_1 x = -1$$
$$\lambda_1 y = 5/2$$

so that $y = -5/2\, x$. Substituting this into $g(x,y,z) = 0$ gives:

$$x^2 + \frac{25}{4} x^2 = 1$$
$$x = \pm\sqrt{4/29}.$$

$z$ can be computed using $z = 1 - x + y$.

| $x$ | $y$ | $z$ | $f(x,y,z)$ | Comments |
|---|---|---|---|---|
| $\sqrt{4/29}$ | $-\frac{5}{2}\sqrt{4/29}$ | $1-\frac{7}{2}\sqrt{4/29}$ | $3-\frac{29}{2}\sqrt{4/29}$ | Global min |
| $-\sqrt{4/29}$ | $\frac{5}{2}\sqrt{4/29}$ | $1+\frac{7}{2}\sqrt{4/29}$ | $3+\frac{29}{2}\sqrt{4/29}$ | Global max. |

MAXIMUM VALUE OF $f(x, y, z)$: $3 - \sqrt{29}$

MINIMUM VALUE OF $f(x, y, z)$: $3 + \sqrt{29}$