MATH 259 – THIRD UNIT TEST

Thursday, April 30, 2009.

NAME: SOLUTIONS

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A	C	D	G	H
B	F	E		

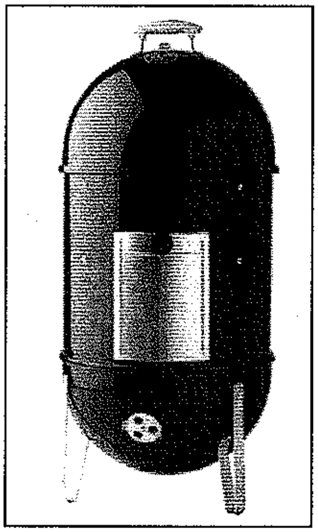
Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	18	
2	16	
3	18	
4	16	
5	16	
6	16	
Total	100	

SOLUTIONS

1. 18 Points. CLEARLY INDICATE YOUR ANSWERS.



The Weber Smokey Mountain smoker (see picture) is a charcoal smoker for cooking meat and other food. The smoker has a cylindrical middle section. The top and bottom of the smoker are shaped like hemispheres. When all measurements are made in units of inches, the volume of the smoker is the same as the volume enclosed by the following three surfaces:

- $x^2 + y^2 + (z-24)^2 = 144$ where $z \geq 24$.
- $x^2 + y^2 = 144$
- $x^2 + y^2 + z^2 = 144$ where $z \leq 0$.

(a) **(6 points)** Write down a **triple integral** that will give the volume of the smoker using **Cartesian coordinates**. You do not have to evaluate this integral.

$$\text{Volume} = \int_{-12}^{12} \int_{-\sqrt{144-x^2}}^{\sqrt{144-x^2}} \int_{-\sqrt{144-x^2-y^2}}^{24+\sqrt{144-x^2-y^2}} 1 \, dz \, dy \, dx$$

or $$\text{Volume} = \int_{-12}^{12} \int_{-\sqrt{144-y^2}}^{\sqrt{144-y^2}} \int_{-\sqrt{144-x^2-y^2}}^{24+\sqrt{144-x^2-y^2}} 1 \, dz \, dx \, dy$$

(b) **(7 points)** Write down a **triple integral** that will give the volume of the smoker using **cylindrical coordinates**. You do not have to evaluate this integral.

$$\text{Volume} = \int_0^{2\pi} \int_0^{12} \int_{-\sqrt{144-r^2}}^{24+\sqrt{144-r^2}} 1 \, r \, dz \, dr \, d\theta$$

or $$\text{Volume} = \int_0^{12} \int_0^{2\pi} \int_{-\sqrt{144-r^2}}^{24+\sqrt{144-r^2}} 1 \, r \, dz \, d\theta \, dr$$

(c) **(5 points)** Using your answers from above (or by any other method you know) find the volume of the smoker. Include appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$\text{Volume} = \underbrace{\frac{4}{3}\pi(12)^3}_{\text{sphere of radius 12}} + \underbrace{\pi(12^2)(24)}_{\text{cylinder of radius 12 and height 24}}$$

$$= 5760\pi$$

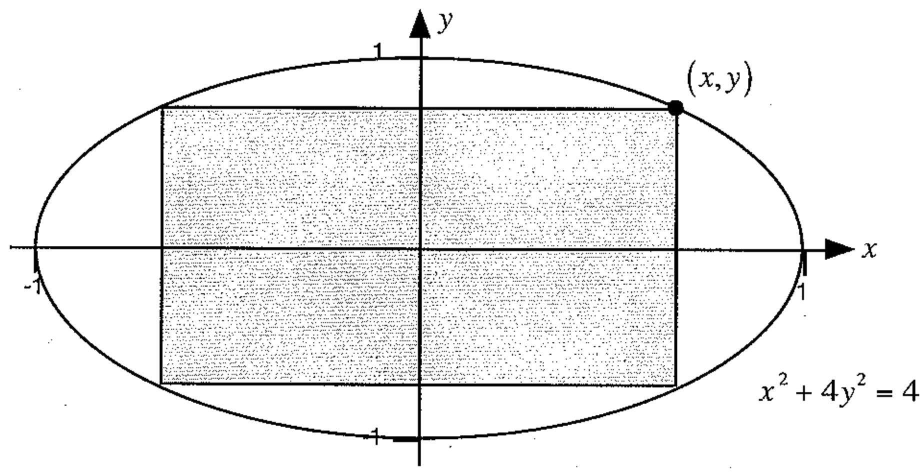
$$\approx 18095.5737 \text{ cubic inches.}$$

2. 16 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

A rectangular wooden beam is cut from a log. The cross-section of the log has the shape of an ellipse (see diagram below). When measurements are made in feet, the equation of the ellipse is:

$$x^2 + 4y^2 = 4.$$

For a beam cut in the way shown in the diagram (below), the strength of the beam is: $S(x, y) = 4xy$. Find the values of x and y that maximize the strength of the rectangular beam.



Show your work and record your answers in the spaces provided on the next page.

There are many ways to solve this problem. We will use Lagrange multipliers to maximize $S(x,y) = 4xy$ subject to the constraint:

$$g(x,y) = x^2 + 4y^2 - 4 = 0.$$

$\nabla S = \langle 4y, 4x \rangle \qquad \nabla g = \langle 2x, 8y \rangle.$

The Lagrange multiplier system of equations is:

$\nabla S = \lambda \nabla g \qquad$ or: $\qquad 4y = 2\lambda x$

$\qquad\qquad\qquad\qquad\qquad\quad 4x = 8\lambda y$

Additional space for your work is provided on the next page.

SOLUTIONS

SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

A rectangular wooden beam is cut from a log. The cross-section of the log has the shape of an ellipse (see previous page). When measurements are made in feet, the equation of the ellipse is:

$$x^2 + 4y^2 = 4.$$

For a beam cut in the way shown in the diagram (see previous page), the strength of the beam is: $S(x, y) = 4xy$. Find the values of x and y that maximize the strength of the rectangular beam. Show your work and record your answers in the spaces provided below.

Multiply the first equation by x and the second by y to obtain:

$$2\lambda x^2 = 4xy = 8\lambda y^2.$$

$\lambda \neq 0$ so $x^2 = 4y^2$. Substituting this into the constraint gives:

$$x^2 + x^2 = 4 \quad \text{or} \quad x = \pm\sqrt{2}$$

and $y = \pm \frac{1}{2} x = \pm \frac{1}{\sqrt{2}}$.

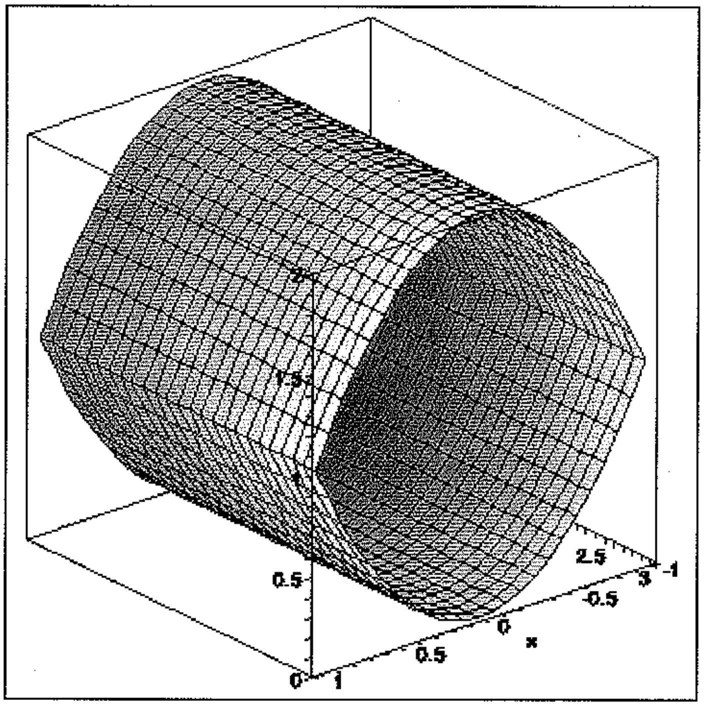
x	y	S(x, y)	Comments
$\sqrt{2}$	$1/\sqrt{2}$	4	Global max.
$\sqrt{2}$	$-1/\sqrt{2}$	-4	
$-\sqrt{2}$	$1/\sqrt{2}$	-4	
$-\sqrt{2}$	$-1/\sqrt{2}$	4	

VALUE OF x THAT MAXIMIZES STRENGTH = $\sqrt{2}$ **feet**

VALUE OF y THAT MAXIMIZES STRENGTH = $1/\sqrt{2}$ **feet**

SOLUTIONS

3. **18 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**



The diagram given on the left shows the solid S bounded by the four surfaces:

- $z = 2 - x^2$
- $z = x^2$
- $y = 0$
- $y = 3.$

In this problem you may assume that all measurements are in units of meters.

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(a) **(8 points)** The solid S is composed of a material whose density is given by:

$$\delta(x,y,z) = 1 + x + y \text{ kg/m}^3.$$

Calculate the mass of the solid S. Clearly indicate your final answer. Include appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$\text{Mass} = \int_0^3 \int_{-1}^1 (1+x+y)(2-2x^2)\,dx\,dy$$

$$= \int_0^3 \int_{-1}^1 (2 - 2x^2 + 2x - 2x^3 + 2y - 2x^2y)\,dx\,dy$$

$$= \int_0^3 \left[2x - \frac{2}{3}x^3 + x^2 - \frac{1}{2}x^4 + 2xy - \frac{2}{3}x^3y\right]_{-1}^1 dy$$

$$= \int_0^3 \frac{8}{3}(1+y)\,dy$$

$$= \left[\frac{8}{3}\left(y + \frac{1}{2}y^2\right)\right]_0^3 = 20 \text{ kg}$$

Continued on the next page.

SHOW YOUR WORK. NO WORK = NO CREDIT.

The diagram given on the previous page shows the solid S bounded by the four surfaces:

- $z = 2 - x^2$
- $z = x^2$
- $y = 0$
- $y = 3.$

In this problem you may assume that all measurements are in units of meters.

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(b) **(10 points)** Find the value of $\bar{y}$, the y-coordinate of the center of mass of S. Clearly indicate your final answer. Include appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$\bar{y} = \frac{\int_0^3 \int_{-1}^1 y(1+x+y)(2-2x^2)\,dx\,dy}{20}$$

$$= \frac{\int_0^3 \frac{8}{3}(y+y^2)\,dy}{20}$$

$$= \frac{\left[\frac{8}{3}\left(\frac{1}{2}y^2+\frac{1}{3}y^3\right)\right]_0^3}{20}$$

$$= \frac{36}{20}$$

$$= 1.8000 \text{ m.}$$

SOLUTIONS

4. 16 Points. SHOW YOUR WORK. CLEARLY INDICATE YOUR ANSWERS.

In this problem, the path C is part of the graph of $y^2 = x^3$ from the point $(1, 1)$ to the point $(4, 8)$. Calculate the exact value of the path integral:

$$\int_C y \cdot \sqrt{x} \cdot dx + x \cdot \sqrt{x} \cdot dy.$$

Clearly indicate your final answer. If you give your answer as a decimal, include at least four (4) decimal places.

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

First we must find parametric equations for x and y.

$$x(t) = t \qquad 1 \le t \le 4.$$
$$y(t) = t^{3/2}$$

Then:

$$\int_c y \cdot \sqrt{x}\, dx = \int_1^4 t^{3/2} \cdot t^{1/2} \cdot (1) \cdot dt$$

$$= \int_1^4 t^2\, dt$$

$$= \left[\tfrac{1}{3} t^3\right]_1^4 = 21$$

$$\int_c x \cdot \sqrt{x}\, dy = \int_1^4 t \cdot t^{1/2} \cdot \tfrac{3}{2} \cdot t^{1/2} \cdot dt$$

$$= \int_1^4 \tfrac{3}{2} t^2\, dt = \frac{63}{2}$$

Finally:

$$\int_c y \cdot \sqrt{x}\, dx + x \cdot \sqrt{x}\, dy = 21 + \frac{63}{2} = \frac{105}{2}$$

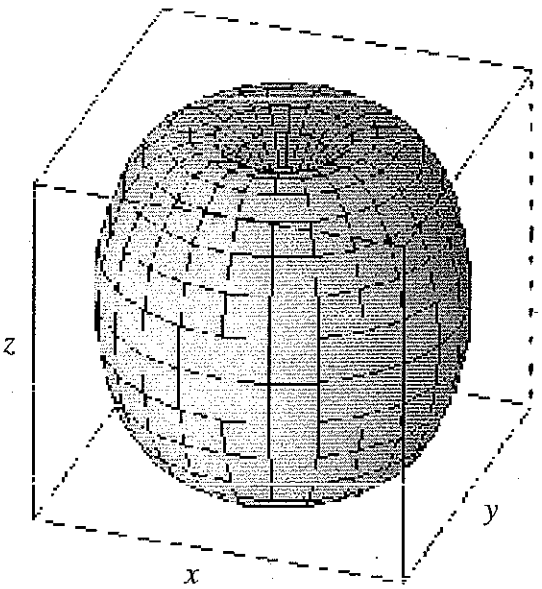
$$= 52.5000$$

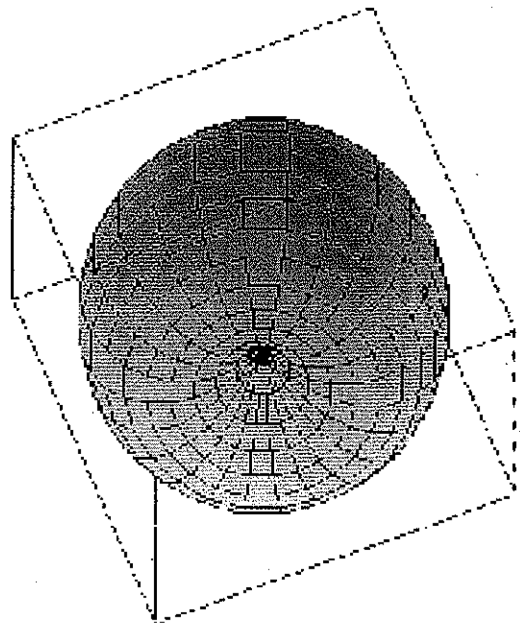
5. 16 Points. SHOW YOUR WORK. CLEARLY INDICATE YOUR FINAL ANSWER.

The diagram shown below is the surface (from two different points of view) whose equation in spherical coordinates is:

$$\rho = 2a \cdot \sin(\varphi),$$

where $a > 0$ is a constant. Find the volume enclosed by this surface.





NOTE: **(a)** You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(b) You may use the following antidifferentiation formula without justifying it:

$$\int \sin^4(x)dx = \tfrac{-1}{4}\sin^3(x)\cos(x) - \tfrac{3}{8}\cos(x)\sin(x) + \tfrac{3}{8}x + C.$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{2a\cdot\sin(\varphi)} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{1}{3}\rho^3 \cdot \sin(\varphi)\right]_0^{2a\cdot\sin(\varphi)} d\varphi\, d\theta$$

$$= \frac{8a^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin^4(\varphi)\, d\varphi\, d\theta$$

Additional space for your work is provided on the next page.

SHOW YOUR WORK. CLEARLY INDICATE YOUR FINAL ANSWER.

The diagram shown on the previous page is the surface (from two different points of view) whose equation in spherical coordinates is:

$$\rho = 2a \cdot \sin(\varphi),$$

where $a > 0$ is a constant. Find the volume enclosed by this surface.

NOTE: **(a)** You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(b) You may use the following antidifferentiation formula without justifying it:

$$\int \sin^4(x)dx = \tfrac{-1}{4}\sin^3(x)\cos(x) - \tfrac{3}{8}\cos(x)\sin(x) + \tfrac{3}{8}x + C.$$

$$\text{Volume} = \frac{8a^3}{3}\int_0^{2\pi}\left[\frac{-1}{4}\sin^3(\varphi)\cos(\varphi) - \frac{3}{8}\cos(\varphi)\sin(\varphi) + \frac{3}{8}\varphi\right]_0^{\pi} d\theta$$

$$= \pi a^3 \int_0^{2\pi} d\theta$$

$$= 2\pi^2 a^3$$

6. 16 Points. SHOW YOUR WORK.

The intersection of the cylinder:

$$x^2 + y^2 = 1,$$

and the plane:

$$2x - y + z = 4,$$

is an ellipse. Find the coordinates (x, y and z) of the **highest** and **lowest** points on this ellipse. Record your final answers in the spaces provided on the next page.

We will use Lagrange multipliers to find the global maximum and minimum of:

$$f(x, y, z) = z$$

subject to the constraints:

$$g(x, y, z) = x^2 + y^2 - 1 = 0$$

$$h(x, y, z) = 2x - y - z - 4 = 0.$$

$$\nabla f = \langle 0, 0, 1 \rangle$$

$$\nabla g = \langle 2x, 2y, 0 \rangle$$

$$\nabla h = \langle 2, -1, 1 \rangle$$

The Lagrange Multiplier system of equations $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$ gives:

$$0 = 2\lambda_1 x + 2\lambda_2$$

$$0 = 2\lambda_1 y - \lambda_2$$

$$1 = \lambda_2.$$

Since $\lambda_2 = 1$ we can rewrite the first two

Additional space for your work is provided on the next page.

SOLUTIONS

SHOW YOUR WORK.

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and the plane:

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is an ellipse. Find the coordinates (x, y and z) of the **highest** and **lowest** points on this ellipse. Record your final answers in the spaces provided below.

equations as: $\lambda_1 x = -1$

$\lambda_1 y = 1/2$

so that $x = -2y$. Substituting this into the first constraint gives:

$(2y)^2 + y^2 = 1$ so that $y = \pm 1/\sqrt{5}$ and $x = \pm 2/\sqrt{5}$.

We will substitute these points into the second constraint to find z as $z = 4 - 2x + y$.

$x = 2/\sqrt{5}$ $y = -1/\sqrt{5}$: $z = 4 - 4/\sqrt{5} - \frac{1}{\sqrt{5}}$

$x = -2/\sqrt{5}$ $y = 1/\sqrt{5}$: $z = 4 + 4/\sqrt{5} + 1/\sqrt{5}$

Point	x	y	z
Highest point on ellipse	$-2/\sqrt{5}$	$-1/\sqrt{5}$	$4+\sqrt{5}$
Lowest point on ellipse	$2/\sqrt{5}$	$1/\sqrt{5}$	$4-\sqrt{5}$