MATH 259 – SECOND UNIT TEST

Tuesday, March 31, 2009.

NAME: SOLUTIONS

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A	C	D	G	H
B	F	E		

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	15	
2	16	
3	23	
4	13	
5	16	
6	17	
Total	100	

1. **15 Points. CLEARLY INDICATE YOUR ANSWERS.**

(a) **(7 points)** The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

$x = 30 \qquad dx = 0.1 \qquad y = 24 \qquad dy = 0.1$

$A = x \cdot y \qquad dA = y \cdot dx + x \cdot dy$

$= (24)(0.1) + (30)\cdot(0.1)$

$= 5.4 \text{ cm}^2$

(b) **(8 points)** Let $f(x, y)$ be a differentiable function. Consider the points listed below.

$A = (1, 3) \qquad B = (3, 3) \qquad C = (1, 7) \qquad D = (6, 15).$

Let $\vec{v}$ be the vector that stretches from A to B and let $\vec{w}$ be the vector that stretches from A to C. Then:

$D_{\vec{v}} f(1,3) = 3$ and $D_{\vec{w}} f(1,3) = 26.$

Let $\vec{q}$ be the vector that stretches from A to D. Find $D_{\vec{q}} f(1,3)$.

Unit vector from A to B: $\vec{u}_1 = \langle 1, 0 \rangle.$

Unit vector from A to C: $\vec{u}_2 = \langle 0, 1 \rangle.$

$\nabla f(1,3) = \langle a, b \rangle.$ Then:

$3 = D_{\vec{v}} f(1,3) = \nabla f(1,3) \cdot \vec{u}_1 = \langle a, b \rangle \cdot \langle 1, 0 \rangle$

$= a.$

$26 = D_{\vec{w}} f(1,3) = \nabla f(1,3) \cdot \vec{u}_2 = \langle a, b \rangle \cdot \langle 0, 1 \rangle$

$= b.$

Unit vector from A to D: $\vec{u}_3 = \langle 5/13, 12/13 \rangle.$

$D_{\vec{q}} f(1,3) = \langle 3, 26 \rangle \cdot \langle 5/13, 12/13 \rangle$

$= 327/13 \approx 25.15384615$

SOLUTIONS

2. 16 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

In this problem you will be concerned with the function:

$$f(x,y) = x^4 + y^4 - 4xy + 2.$$

(a) **(8 points)** Find the x and y coordinates of all critical points of this function. Record your results in the table provided at the bottom of the next page.

We need to find the coordinates of all points where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 4x^3 - 4y = 0 \\ \frac{\partial f}{\partial y} &= 4y^3 - 4x = 0 \end{aligned}\right\}$$

$x^3 = y \quad \cdots$ ①

$y^3 = x \quad \cdots$ ②

So that:

$x^9 = x$

$y^9 = y$

The possible solutions of these equations are:

$(x,y) = (0,0)$

$(x,y) = (1,1)$

$(x,y) = (-1,-1)$.

Continued on the next page.

SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

In this problem you will be concerned with the function:

$$f(x,y) = x^4 + y^4 - 4xy + 2.$$

(b) **(8 points)** Classify each of the critical points of the function. Record your results in the table provided at the bottom of this page.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -4$$

Evaluating Jacobian determinant:

At $(x,y) = (0,0)$: $\quad (f_{xx})(f_{yy}) - (f_{xy})^2 = (12\cdot 0^2)(12\cdot 0^2) - (-4)^2$
$= -16 < 0 \}$ Saddle.

At $(x,y) = (1,1)$: $\quad (f_{xx})(f_{yy}) - (f_{xy})^2 = (12)(12) - (-4)^2$
$= 144 - 16 > 0$

At $(x,y) = (-1,-1)$: $\quad (f_{xx})(f_{yy}) - (f_{xy})^2 = (12)(12) - (-4)^2$
$= 144 - 16 > 0.$

Evaluating f_{xx}:

$$\left.\begin{array}{ll} \text{At } (x,y) = (1,1): & f_{xx} = 12 > 0 \\ \text{At } (x,y) = (-1,-1): & f_{xx} = 12 > 0 \end{array}\right\} \text{local minimums}$$

x	*y*	**Classification**
0	0	Saddle point
1	1	Local minimum
−1	−1	Local minimum.

3. 23 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

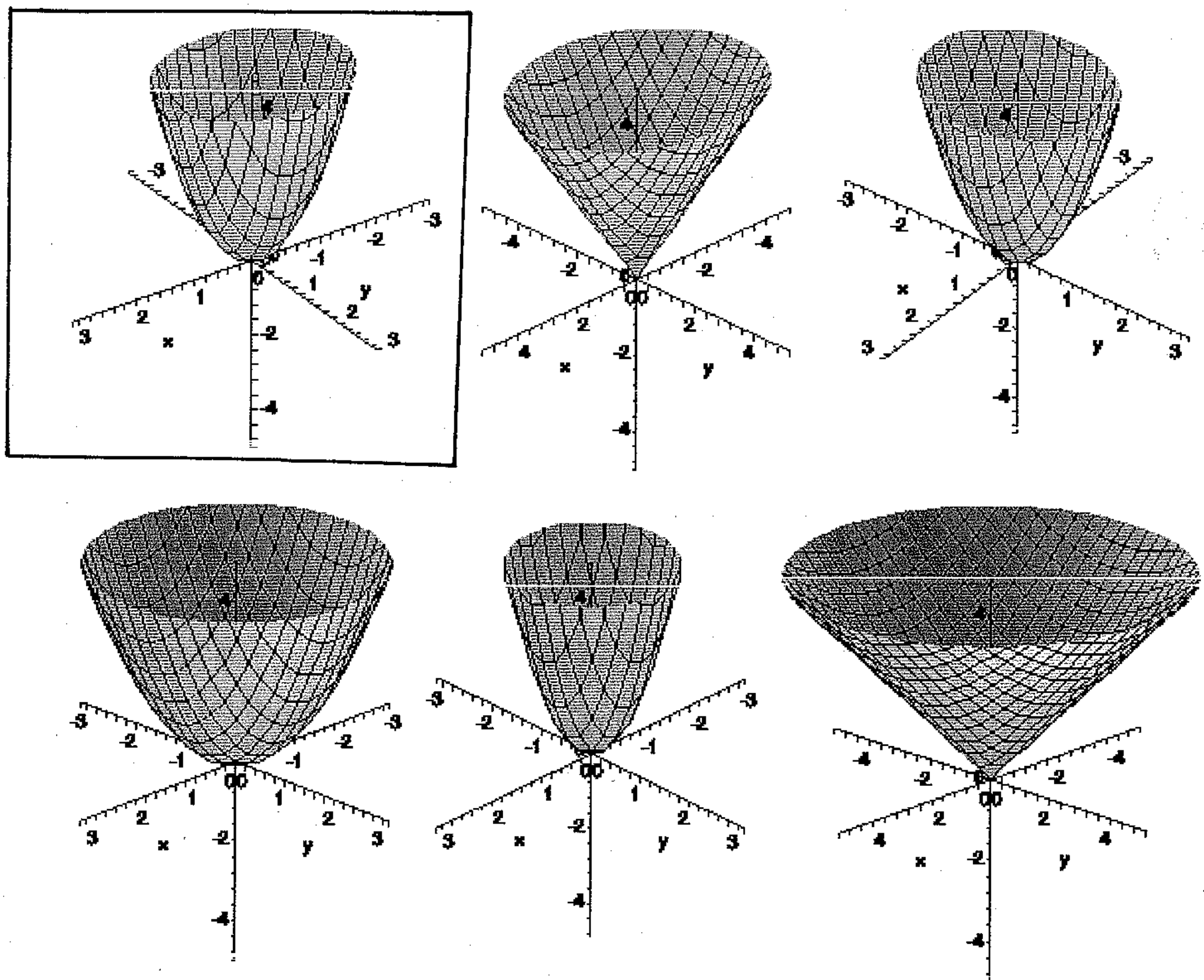
In this problem you will be interested in the surface defined by the function:

$$z = f(x, y) = 2x^2 + 3y^2.$$

(a) **(3 points)** Classify the surface (i.e. is it a plane, ellipsoid, hyperboloid of two sheets, etc.?)

The surface is a parabolic ellipsoid.

(b) **(6 points)** Which graph (of those shown below) does the best job of showing the graph of $z = f(x, y)$? CIRCLE ONE (AND ONLY ONE) GRAPH.



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SHOW YOUR WORK. NO WORK = NO CREDIT.

In this problem you will be interested in the surface defined by the function:

$$z = f(x,y) = 2x^2 + 3y^2.$$

(c) **(6 points)** Find an equation for the tangent plane to $z = f(x, y)$ based at the point $(x, y) = (2, 1)$.

$$z_0 = f(2,1) = 2(2^2) + 3(1^2) = 8 + 3 = 11.$$

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial x}(2,1) = 8$$

$$\frac{\partial f}{\partial y} = 6y \qquad \frac{\partial f}{\partial y}(2,1) = 6$$

Equation of tangent plane:

$$8(x-2) + 6(y-1) - (z-11) = 0.$$

(d) **(8 points)** Find the coordinates (x, y and z) of the point on the surface $z = f(x, y)$ for which the tangent plane is parallel to the plane:

$$4x - 3y - z = 10.$$

Record your answer in the spaces provided below.

The normal vector to the given plane is: $\vec{n} = \langle 4, -3, -1 \rangle$.
For the tangent planes to be parallel, the normal vectors must be parallel, so for some constant k:

$$\langle 4x, 6y, -1 \rangle = k \langle 4, -3, -1 \rangle, \text{ or:}$$

$$\left.\begin{aligned} 4x &= 4k \\ 6y &= -3k \\ -1 &= -k \end{aligned}\right\} \quad \text{so } \begin{aligned} k &= 1 \\ x &= 1 \\ y &= -1/2. \end{aligned}$$

To get z, evaluate:

$$f(1, -\tfrac{1}{2}) = 2(1^2) + 3(-\tfrac{1}{2})^2 = 2\tfrac{3}{4}$$

FINAL ANSWERS:

$x =$ 1 $\qquad y =$ $-1/2$ $\qquad z =$ $11/4$

4. 13 Points. CLEARLY INDICATE YOUR ANSWERS.

In this problem, the temperature at a point (x, y, z) is given by the function:

$$W(x,y,z) = 50 + xyz,$$

where temperature is measure in degrees Celsius (°C) and x, y and z are all measured in meters.

(a) **(5 points)** Find the rate of change of temperature that a person would experience if they started at the point (3, 4, 1) and moved in the direction of the vector $\vec{v} = \langle 1,2,2 \rangle$. Include appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$\nabla W = \langle yz, xz, xy \rangle \qquad \nabla W(3,4,1) = \langle 4, 3, 12 \rangle$$

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 2^2} = 3 \qquad \vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \langle 1/3, 2/3, 2/3 \rangle.$$

$$D_{\vec{v}} W(3,4,1) = \langle 4, 3, 12 \rangle \bullet \langle 1/3, 2/3, 2/3 \rangle$$

$$= 4/3 + 6/3 + 24/3$$

$$= 34/3 \ {}^\circ C / m.$$

(b) **(4 point)** If the person is standing at the point (3, 4, 1), in what direction should they move to experience the greatest rate of change of temperature? Give your answer in the form of a **vector**.

$$\text{Direction} = \nabla W(3,4,1) = \langle 4, 3, 12 \rangle.$$

(c) **(4 point)** If the person is standing at the point (3, 4, 1), what is the greatest rate of change of temperature that they could possibly experience? Give appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$\text{Maximum rate of change} = |\nabla W(3,4,1)| = \sqrt{4^2 + 3^2 + 12^2}$$

$$= 13 \ {}^\circ C/m$$

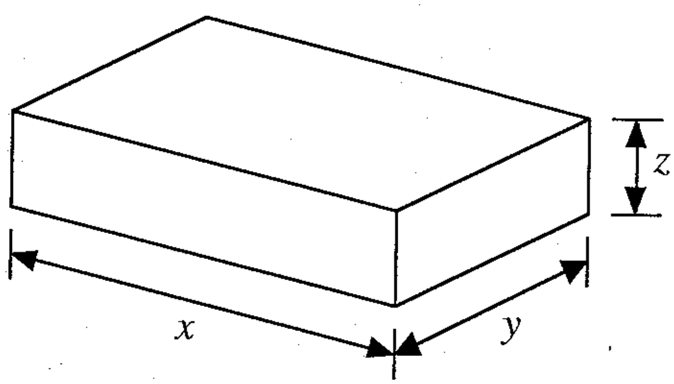
SOLUTIONS

5. 16 Points. NO PARTIAL CREDIT WITHOUT WORK.

A box is built in the shape of a rectangular box (see below). The US Postal Service will only deliver a package if the sum of height and girth of the package does not exceed 108 inches.

Girth is the perimeter of the side of the box that is perpendicular to the length.

Find the maximum volume of the box that the US Postal Service will deliver. Record your final answer in the space provided at the bottom of the next page. Include appropriate units with your answer.



Volume:

$$V = x \cdot y \cdot z$$

Constraint:

$$x + 2y + 2z = 108$$

Rearrange the constraint to get:

$$x = 108 - 2y - 2z$$

and substitute for x in the formula for V:

$$V = y \cdot z\,(108 - 2y - 2z)$$

$$= 108 \cdot y \cdot z - 2 \cdot y^2 \cdot z - 2 \cdot y \cdot z^2.$$

To find critical point(s):

$$\frac{\partial V}{\partial y} = 108z - 4 \cdot y \cdot z - 2 \cdot z^2 = 0 \quad \ldots \text{①}$$

$$\frac{\partial V}{\partial z} = 108y - 2y^2 - 4 \cdot y \cdot z = 0 \quad \ldots \text{②}$$

Additional space for your work is provided on the next page.

SOLUTIONS

A box is built in the shape of a rectangular box (see below). The US Postal Service will only deliver a package if the sum of height and girth of the package does not exceed 108 inches.

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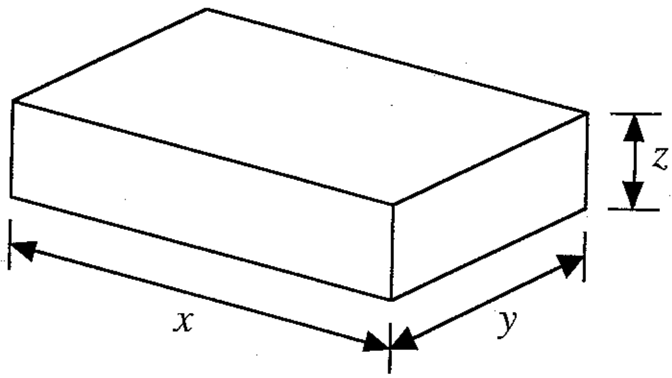

$$\frac{\partial^2 V}{\partial y^2} = -4z$$

$$\frac{\partial^2 V}{\partial z^2} = -4y$$

$$\frac{\partial^2 V}{\partial y\, \partial z} = 108 - 4y - 4z$$

From equation ①: $2z(54 - 2y - z) = 0 \quad \ldots ③$

From equation ②: $2y(54 - y - 2z) = 0 \quad \ldots ④$

Case 1: $z = 0$ Then $y = 0$ or $y = 54$.

Case 2: $y = 0$ Then $z = 0$ or $z = 54$.

Case 3: $y \neq 0$ and $z \neq 0$: $\left.\begin{aligned} 2y + z &= 54 \\ y + 2z &= 54 \end{aligned}\right\}$ $y = z = \frac{54}{3}$

y	z	$D = V_{yy}V_{zz} - (V_{yz})^2$	V_{yy}
0	0	-108^2	N/A
0	54	-108^2	N/A
54	0	-108^2	N/A
54/3	54/3	3888	72

FINAL ANSWER:

Volume = $(18)(18)(36) = 11\,664$ cubic inches.

6. 17 Points. SHOW YOUR WORK AND EXPLAIN YOUR REASONING.

(a) (8 points) Show that the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x \cdot y}{x^2 + y^2}$$

does NOT exist. Be careful to explain WHY the calculations that you perform SHOW that the limit does not exist.

There are many strategies that may be used here. We use the $y = m \cdot x$ substitution strategy.

Substitute $y = m \cdot x$ into $\dfrac{x \cdot y}{x^2 + y^2}$ to obtain:

$$\frac{x \cdot m \cdot x}{x^2 + (m \cdot x)^2} = \frac{m}{1 + m^2}, \quad x \neq 0.$$

As this result depends on m, it means that as we take the limit of $\dfrac{x \cdot y}{x^2 + y^2}$ by approaching $(0,0)$ along lines of the form $y = m \cdot x$, we will get a different value of the "limit" for each different line we use.

Continued on the next page.

SOLUTIONS

SHOW YOUR WORK AND EXPLAIN YOUR REASONING.

(b) **(9 points)** Use the ε–δ definition of a limit to show the following.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0.$$

Step 1: $|f(x,y) - L| = \left| \frac{x^2-y^2}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x^2-y^2|}{\sqrt{x^2+y^2}}$

Step 2: $\frac{|x^2-y^2|}{\sqrt{x^2+y^2}} \leq \frac{|x|^2+|y|^2}{\sqrt{x^2+y^2}} \leq \frac{\left(\sqrt{x^2+y^2}\right)^2}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}$

Step 3: Let $\varepsilon > 0$ be given. Set $\delta = \varepsilon$.