MATH 259 – FIRST UNIT TEST

Friday, February 13, 2009.

NAME: SOLUTIONS

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A	C	D	G	H
B	F	E		

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	17	
2	18	
3	25	
4	16	
5	16	
6	8	
Total	100	

1. 17 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

An object is moving along a curve in the x-y plane. The position of the object at time t is given by the parametric equations $x(t)$ and $y(t)$. All that you can assume about $x(t)$ and $y(t)$ is that their derivatives are given by:

$$\frac{dx}{dt} = t \cdot \cos\left(t^3\right) \qquad \text{and} \qquad \frac{dy}{dt} = -t \cdot \sin\left(t^3\right)$$

for $0 \le t \le 3$, and that at time $t = 2$ their values are:

- $x(2) = 4$, and,
- $y(2) = 5$.

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(a) **(7 points)** Write down an equation for the tangent line to the curve at the point (4, 5).

$$\left.\frac{dy}{dt}\right|_{t=2} = -2 \cdot \sin(8) \qquad \left.\frac{dx}{dt}\right|_{t=2} = 2 \cdot \cos(8).$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(4,5)} = -\tan(8).$$

Tangent line: $\quad y - 5 = -\tan(8) \cdot (x - 4)$

(b) **(3 points)** Find the speed of the object at time $t = 2$.

$$\text{Speed} = \sqrt{4 \cdot \sin^2(8) + 4 \cdot \cos^2(8)} = \sqrt{4} = 2$$

(c) **(7 points)** Find the **exact** distance that the object travels in the time interval $0 \le t \le 1$.

$$\text{Distance} = \int_0^1 \sqrt{t^2 \cdot \cos^2(t^3) + t^2 \cdot \sin^2(t^3)}\, dt$$

$$= \int_0^1 t \, dt$$

$$= \left[\frac{1}{2} t^2\right]_0^1$$

$$= \frac{1}{2}$$

SOLUTIONS

2. **18 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.**

A plane has the equation $z = 5x - 2y + 7$. $\quad 5x - 2y - z = -7$

(a) **(5 points)** Find a value of λ that makes the vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ perpendicular to the plane.

Normal vector: $\langle 5, -2, -1 \rangle$. $\quad$ Vector: $\langle \lambda, 1, 1/2 \rangle$

$\vec{0} = \langle 5, -2, 1 \rangle \times \langle \lambda, 1, 1/2 \rangle = \langle 0, -\lambda - 5/2, 2\lambda + 5 \rangle$

Solved for: $\lambda = -5/2$

(b) **(5 points)** Find a value of α so that the point $(\alpha+1, \alpha, \alpha-1)$ lies on the plane.

$$\alpha - 1 = 5(\alpha + 1) - 2\alpha + 7$$
$$\alpha - 1 = 3\alpha + 12$$
$$-13 = 2\alpha$$

Solved for: $\alpha = -13/2$.

(c) **(8 points)** Is the plane $z = 5x - 2y + 7$ parallel to the plane $4x + 3y + 2z = -1$ or do the two planes intersect? Either show that the planes are parallel or find a **vector** equation for the line of intersection.

Normal vectors: $\vec{n_1} = \langle 5, -2, -1 \rangle$

$\vec{n_2} = \langle 4, 3, 2 \rangle$

$$\vec{n_1} \times \vec{n_2} = \begin{array}{ccccc} i & j & k & i & j \\ 5 & -2 & -1 & 5 & -2 \\ 4 & 3 & 2 & 4 & 3 \end{array} = \langle -1, -14, 23 \rangle$$

Planes are not parallel. To find a point on the line assume $z = 0$, and solve:

$$\left.\begin{array}{l} 5x - 2y = -7 \\ 4x + 3y = -1 \end{array}\right\} \quad \text{Solution is } x = -1 \quad y = 1$$

So the point on the line is: $(-1, 1, 0)$.

Vector equation for line:

$$\langle x, y, z \rangle = \langle -1, 1, 0 \rangle + t \cdot \langle -1, -14, 23 \rangle$$

SOLUTIONS

3. 25 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

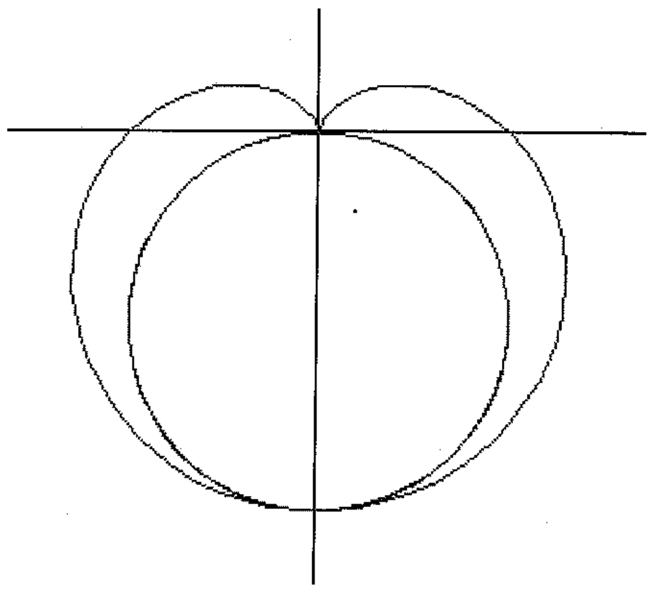
You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(a) **(8 points)** The diagram given below shows the graphs of the two polar curves:

$$r = -2\cdot\sin(\theta) \qquad \text{and} \qquad r = 1-\sin(\theta).$$

Find the **exact** coordinates (x and y) of all intersection points. Show your work and record your results in the table at the bottom of the page. No work = no credit.



$$-2\sin(\theta) = 1-\sin(\theta)$$
$$\sin(\theta) = -1$$
$$\theta = 3\pi/2$$

There is also an intersection at $r = 0$, although the values of θ for which the curves reach this point are different.

x	y
0	0
0	-2

Continued on the next page.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

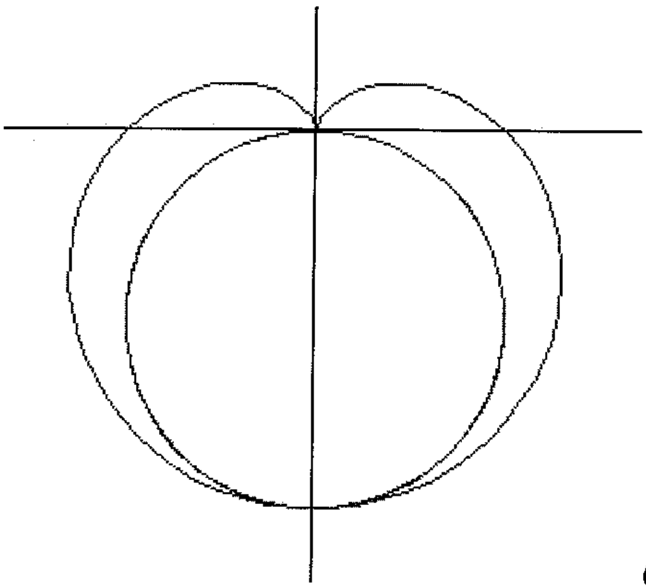
You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(b) **(9 points)** Find the coordinates (x and y) of all points where the tangent line to the polar curve:

$$r = 1 - \sin(\theta)$$

is **horizontal**. Show your work and record your results in the table below. No work = no credit.



$$y = r \cdot \sin(\theta)$$
$$= \sin(\theta) - \sin^2(\theta)$$
$$\frac{dy}{d\theta} = \cos(\theta) - 2\sin(\theta)\cos(\theta)$$
$$= \cos(\theta)\cdot(1 - 2\sin(\theta))$$

$\frac{dy}{d\theta} = 0$ when:

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Of these, note that the tangent line is not horizontal for $\theta = \pi/2$ as $dx/d\theta\,|_{\theta=\pi/2} = 0$.

x	y
0	$-2 \qquad (\theta = 3\pi/2)$
$-\sqrt{3}/4$	$1/4 \qquad (\theta = 5\pi/6)$
$\sqrt{3}/4$	$1/4 \qquad (\theta = \pi/6)$

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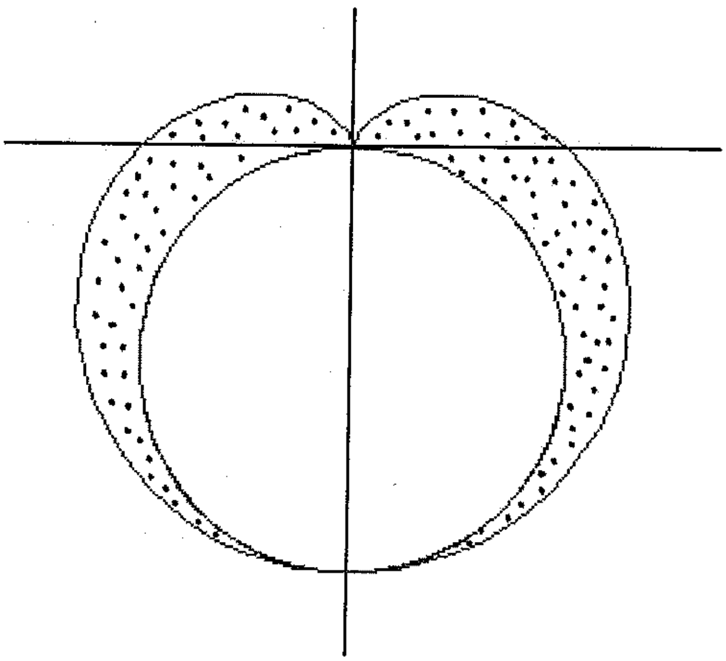
SOLUTIONS

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(c) **(8 points)** Find the **exact** value of the area that is outside of the polar curve $r = -2\cdot\sin(\theta)$ and inside the polar curve $r = 1-\sin(\theta)$. Clearly indicate your final answer and show your work. No work = no credit.



The limits of integration for the outer curve $r = 1 - \sin(\theta)$ are 0 to 2π.

The limits of integration for the inner curve $r = -2\sin(\theta)$ are 0 to π.

$$A = \frac{1}{2}\int_0^{2\pi}(1-\sin(\theta))^2\,d\theta - \frac{1}{2}\int_0^{\pi}(-2\sin(\theta))^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1-2\sin(\theta)+\sin^2(\theta))\,d\theta - \frac{1}{2}\int_0^{\pi}4\sin^2(\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1-2\sin(\theta)+\frac{1}{2}-\cos(2\theta)\right)d\theta - \int_0^{\pi}1-\cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\left[3/2\,\theta + 2\cos(\theta) - \frac{1}{2}\sin(2\theta)\right]_0^{2\pi} - \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi}$$

$$= \frac{3\pi}{2} - \pi$$

$$= \frac{\pi}{2}.$$

4. **16 Points.**

Consider the polar curve defined by the equation:

$$r = \frac{12}{2 + 4 \cdot \sin(\theta)}.$$

(a) **(4 points)** Use the equation given above to complete all entries in the table shown below. If you give your answers as decimals, be sure to give your answers to at least four (4) decimals places.

θ	r	x	y
0	6	6	0
$\pi/2$	2	0	2
π	6	-6	0
$3\pi/2$	-6	0	6

(b) **(3 points)** Find the eccentricity of the curve.

$$r = \frac{12}{2 + 4 \cdot \sin(\theta)} = \frac{6}{1 + 2 \cdot \sin(\theta)}$$

Eccentricity = 2.

(c) **(3 points)** Classify the curve (i.e. is it a line, circle, ellipse, cardioid, etc. ?).

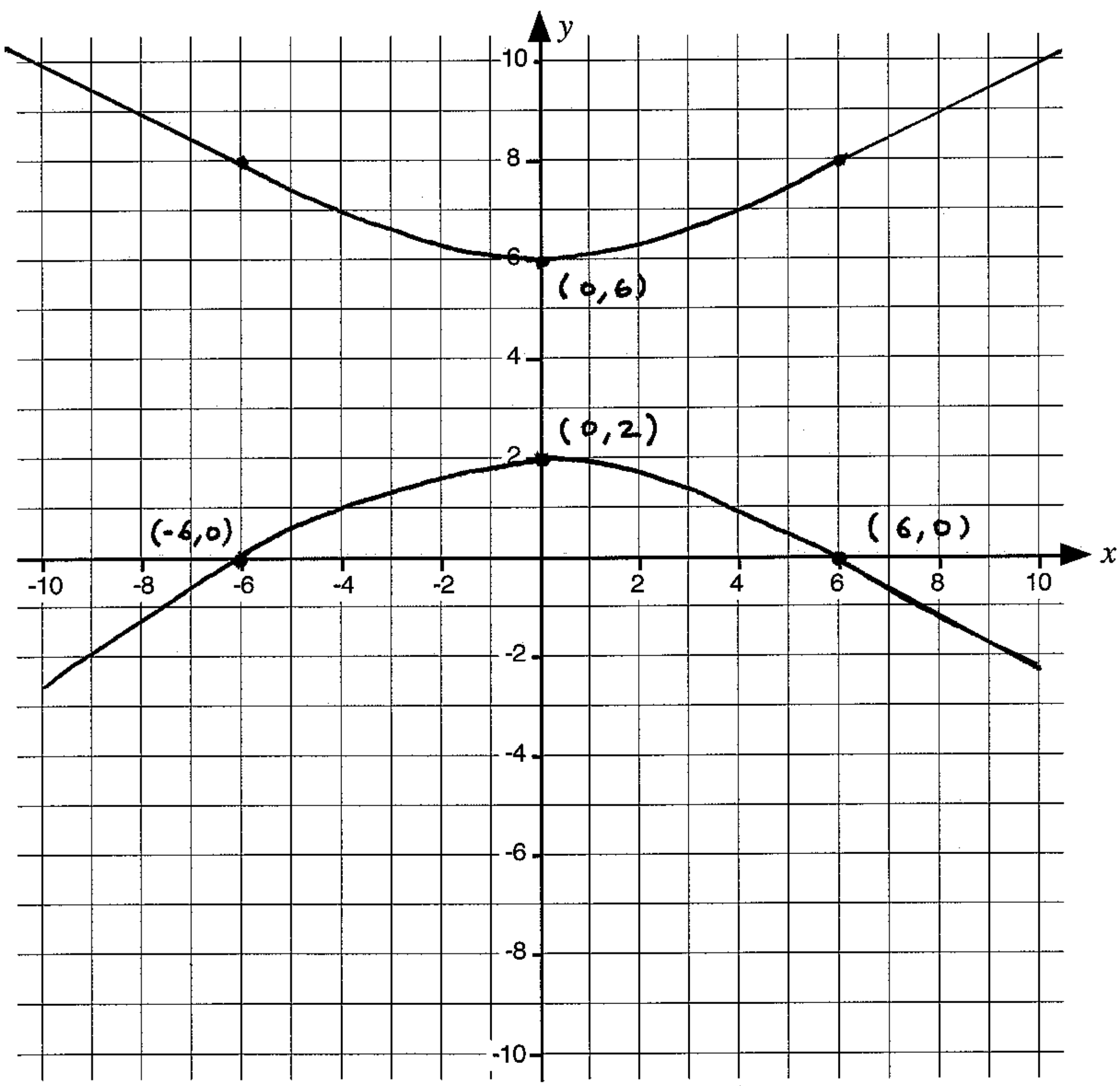
Hyperbola (as eccentricity > 1).

Continued on the next page.

Consider the polar curve defined by the equation:

$$r = \frac{12}{2 + 4 \cdot \sin(\theta)}.$$

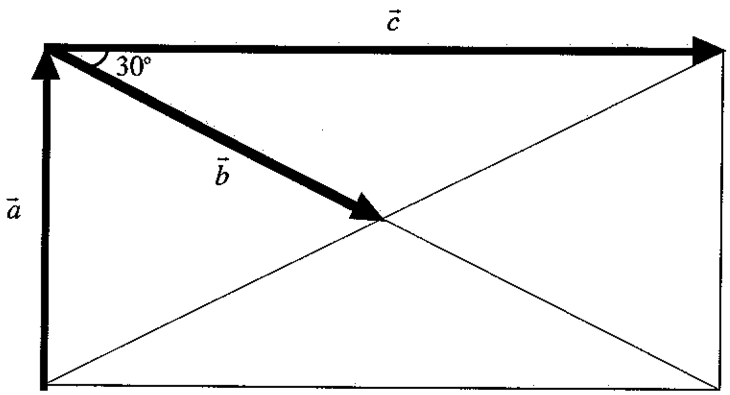
(d) **(6 points)** Use the axes provided below to sketch an accurate graph of the curve $r = \dfrac{12}{2 + 4 \cdot \sin(\theta)}$ in the xy-plane. On your graph, include the coordinates of any points of interest such as x- or y-intercepts.



SOLUTIONS

5. **16 Points. NO PARTIAL CREDIT WITHOUT WORK.**

(a) **(6 points)** Use the diagram provided below to calculate the value of $\vec{a} \bullet \vec{b}$. You can assume that $|\vec{c}| = 2$.



$$\vec{a} \bullet \vec{b} = |\vec{a}|\,|\vec{b}| \cos(60°)$$

$$\frac{|\vec{a}|}{|\vec{c}|} = \tan(30°)$$

$$|\vec{a}| = 2 \cdot \tan(30°) = \frac{2}{\sqrt{3}}$$

Next, $\dfrac{|\vec{c}|}{2|\vec{b}|} = \cos(30°)$ so $|\vec{b}| = \dfrac{1}{\cos(30°)} = \dfrac{2}{\sqrt{3}}$

Finally: $\vec{a} \bullet \vec{b} = \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \left(\frac{-1}{2}\right) = \frac{-2}{3}$.

(b) **(10 points)** Find the equation of the plane that contains all of the following points:

(2, 1, 0) (0, 1, 3) (1, 0, 1).

Let $\vec{a} = \langle 2, 1, 0 \rangle - \langle 1, 0, 1 \rangle = \langle 1, 1, -1 \rangle$

Let $\vec{b} = \langle 0, 1, 3 \rangle - \langle 1, 0, 1 \rangle = \langle -1, 1, 2 \rangle$

<u>Normal vector:</u>

$$\vec{n} = \vec{a} \times \vec{b} = \begin{array}{ccccc} i & j & k & i & j \\ 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 2 & -1 & 1 \end{array} = \langle 3, -1, 2 \rangle$$

<u>Equation of Plane:</u>

$$3(x-2) - 1 \cdot (y-1) + 2(z-0) = 0.$$

SOLUTIONS.

6. 8 Points. CLEARLY INDICATE YOUR FINAL ANSWER.

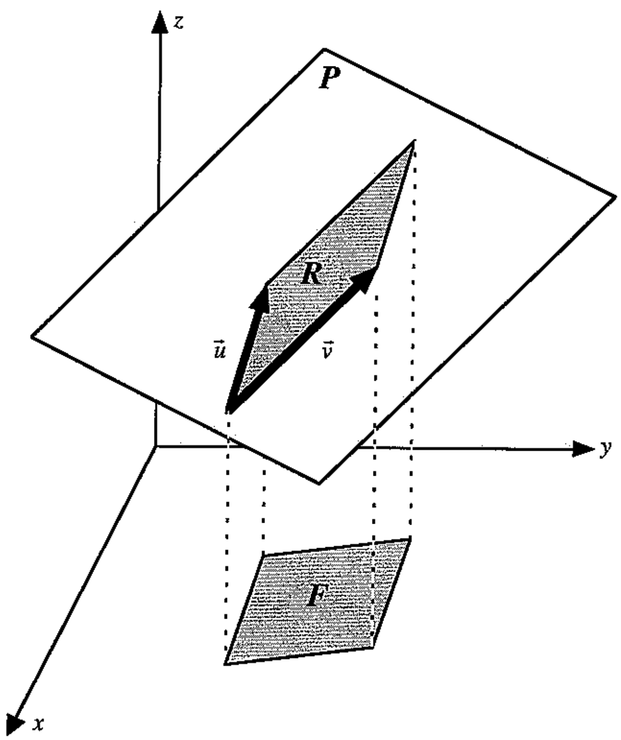
The diagram given below shows part of a plane, P. The equation of this plane is:

$$z = m \cdot x + n \cdot y + c$$

where m and n are both non-zero constants and c is a positive constant. The two vectors:

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \qquad \text{and} \qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

lie in the plane P and form the edges of the parallelogram R. The projection of R onto the xy-plane is another parallelogram, F. Find a formula for the area of F. Clearly indicate your final answer, which may contain m, n, c, u_1, u_2, u_3, v_1, v_2, v_3 and numbers.



The area of F is given by $|\vec{a} \times \vec{b}|$ where $\vec{a}$ and $\vec{b}$ are the projections of $\vec{u}$ and $\vec{v}$ onto the xy-plane.

$$\vec{a} = \langle u_1, u_2, 0 \rangle$$

$$\vec{b} = \langle v_1, v_2, 0 \rangle$$

$$\vec{a} \times \vec{b} = \begin{array}{ccccc} i & j & k & i & j \\ u_1 & u_2 & 0 & u_1 & u_2 \\ v_1 & v_2 & 0 & v_1 & v_2 \end{array} = \langle 0, 0, u_1 v_2 - v_1 u_2 \rangle$$

Area of F $= |\vec{a} \times \vec{b}| = |u_1 v_2 - v_1 u_2|$.