MATH 259 – FINAL EXAM

Friday, May 8, 2009.

NAME: SOLUTIONS

Alexandra Klimova	Oleksii Mostovyi	Reshma Ramadurai	Stephen Russel	William Boney
A	C	D	G	H
B	F	E		

Instructions:

1. Do not separate the pages of the exam.
2. Please read the instructions for each individual question carefully.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Be sure to use appropriate algebraic and limit notation.
7. **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	11	
2	8	
3	10	
4	10	
5	10	
6	10	
7	8	
8	7	
9	7	
10	10	
11	9	
Total	**100**	

SOLUTIONS

1. 11 points.

In this problem the curve that you will be working with is the curve described by the vector function:

$$\vec{r}(t) = \langle 5\cdot\sin(t), 5\cdot\cos(t), 12t\rangle \qquad \text{for } t \geq 0.$$

If you give any of your answers as decimals, include at least four (4) decimal places.

(a) **(2 points)** Find the **unit** tangent vector to the curve at the point $(5, 0, 6\pi)$.

$\vec{r}\,'(t) = \langle 5\cos(t), -5\sin(t), 12\rangle$

At the point $(5, 0, 6\pi)$, $t = \pi/2$.

$\vec{r}\,'(\pi/2) = \langle 0, -5, 12\rangle$

$|\vec{r}\,'(\pi/2)| = \sqrt{(-5)^2 + (12)^2} = 13.$

Unit tangent vector $= \langle 0, -5/13, 12/13\rangle$

(b) **(3 points)** Find the equation of the tangent line to the curve at the point $(0, -5, 12\pi)$.

At the point $(0, -5, 12\pi)$, $t = \pi$.

$\vec{r}\,'(\pi) = \langle -5, 0, 12\rangle$

Equation of tangent line:

$\langle x, y, z\rangle = \langle 0, -5, 12\pi\rangle + t\cdot\langle -5, 0, 12\rangle.$

Continued on the next page.

In this problem the curve that you will be working with is the curve described by the vector function:

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If you give any of your answers as decimals, include at least four (4) decimal places.

(c) **(3 points)** Find the length of the curve between the points $(5, 0, 6\pi)$ and $(0, -5, 12\pi)$.

$$\text{Length of curve} = \int_{\pi/2}^{\pi} |\vec{r}\,'(t)|\,dt$$

$$= \int_{\pi/2}^{\pi} \sqrt{25\cos^2(t) + 25\sin^2(t) + 144}\;dt$$

$$= \int_{\pi/2}^{\pi} \sqrt{169}\;dt$$

$$= \frac{13\pi}{2}$$

(d) **(3 points)** What are the coordinates (x, y, z) of the point that lies a distance of 26π along the curve from the point $(0, 5, 0)$?

At the point $(0, 5, 0)$ $t = 0$. We want to solve the following equation for T:

$$26\pi = \int_0^T |\vec{r}\,'(t)|\,dt = 13\cdot(T - 0)$$

So, $T = 2\pi$ and:

$$(x, y, z) = (5\sin(2\pi),\ 5\cos(2\pi),\ 12(2\pi))$$

$$= (0, 5, 24\pi).$$

SOLUTIONS

2. 8 Points. CLEARLY INDICATE YOUR ANSWERS.

In this problem, the temperature at a point (x, y, z) is given by the function:

$$W(x,y,z) = 100 - x^2 - y^2 - z^2,$$

where temperature is measure in degrees Celsius (°C) and x, y and z are all measured in meters.

(a) **(4 points)** Find the rate of change of temperature that a person would experience if they started at the point (3, –4, 5) and moved in the direction of the vector $\vec{v} = \langle 3,-4,12 \rangle$. Include appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

Unit vector: $|\vec{v}| = \sqrt{3^2 + (-4)^2 + (12)^2} = 13.$

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \langle 3/13, -4/13, 12/13 \rangle.$$

$$\nabla W = \langle -2x, -2y, -2z \rangle$$

$$\nabla W(3,-4,5) = \langle -6, 8, -10 \rangle$$

$$D_{\vec{u}} W(3,-4,5) = \langle -6, 8, -10 \rangle \cdot \langle 3/13, -4/13, 12/13 \rangle$$

$$= \frac{-170}{13} \approx -13.0769 \text{ °C/m}$$

(b) **(2 points)** If the person is standing at the point (3, –4, 5), in what direction should they move to experience the greatest rate of change of temperature? Give your answer in the form of a **vector**.

$$\nabla W(3,-4,5) = \langle -6, 8, -10 \rangle.$$

(c) **(2 points)** If the person is standing at the point (3, –4, 5), what is the greatest rate of change of temperature that they could possibly experience? Give appropriate units with your answer. If you give your answer as a decimal, include at least four (4) decimal places.

$$|\nabla W(3,-4,5)| = \sqrt{(-6)^2 + 8^2 + (-10)^2}$$

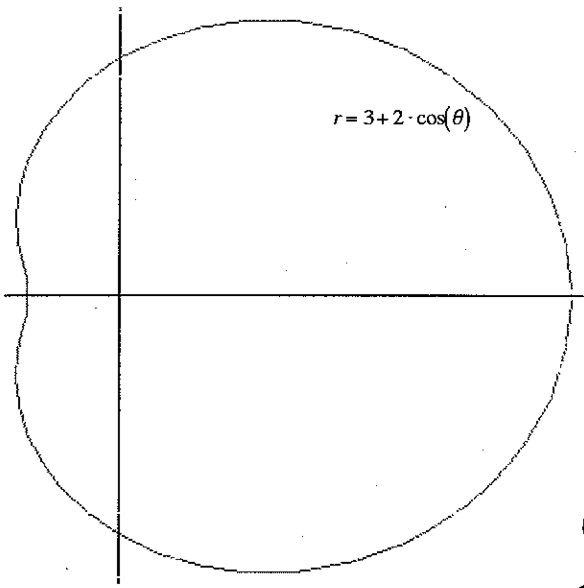
$$\approx +14.1421 \text{ °C/m}.$$

3. **10 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**

Consider the curve defined by the polar equation:

$$r = 3 + 2 \cdot \cos(\theta).$$

(a) **(7 points)** Find the coordinates (x and y) of all points where the tangent line to the polar curve is **horizontal**. Find the **exact** coordinates (x and y) of **all** points where the tangent line is horizontal. Show your work and record your results in the table at the bottom of the page. No work = no credit.



$x(\theta) = 3\cos(\theta) + 2\cos^2(\theta)$

$y(\theta) = 3\sin(\theta) + 2\sin(\theta)\cos(\theta)$

We want all points where $y'(\theta) = 0$ and $x'(\theta) \neq 0$.

$y'(\theta) = 3\cos(\theta) + 2\cos^2(\theta) - 2\sin^2(\theta)$

$= 3\cos(\theta) + 4\cos^2(\theta) - 2 = 0.$

Using the Quadratic formula to solve for $\cos(\theta)$ gives:

$$\cos(\theta) = \frac{-3 \pm \sqrt{9+32}}{8} = 0.4253905297 \text{ and } -1.17539053$$

$\theta = \cos^{-1}(0.4253905297) = 1.131402965$

x	y
1.638085795	3.484994935
1.638085795	−3.484994935

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SOLUTIONS

Consider the curve defined by the parametric equations:

$$x(t) = e^{2t} + e^{-2t} \quad \text{and} \quad y(t) = 3e^{2t} - e^{-2t}.$$

(b) **(3 points)** Find the coordinates (x and y) of **all** points where the tangent line to the parametric curve is **vertical**. Find the **exact** coordinates (x and y) of **all** points where the tangent line is vertical. Show your work and record your results in the table below. No work = no credit.

We want the points where $\frac{dy}{dx}$ is undefined, i.e.

where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$$\frac{dx}{dt} = 2e^{2t} - 2e^{-2t} = 0$$

$$e^{2t} = e^{-2t} \quad \text{so} \quad t = 0$$

When $t = 0$, $x(0) = 2$

$y(0) = 2$.

x	y
2	2

4. 10 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

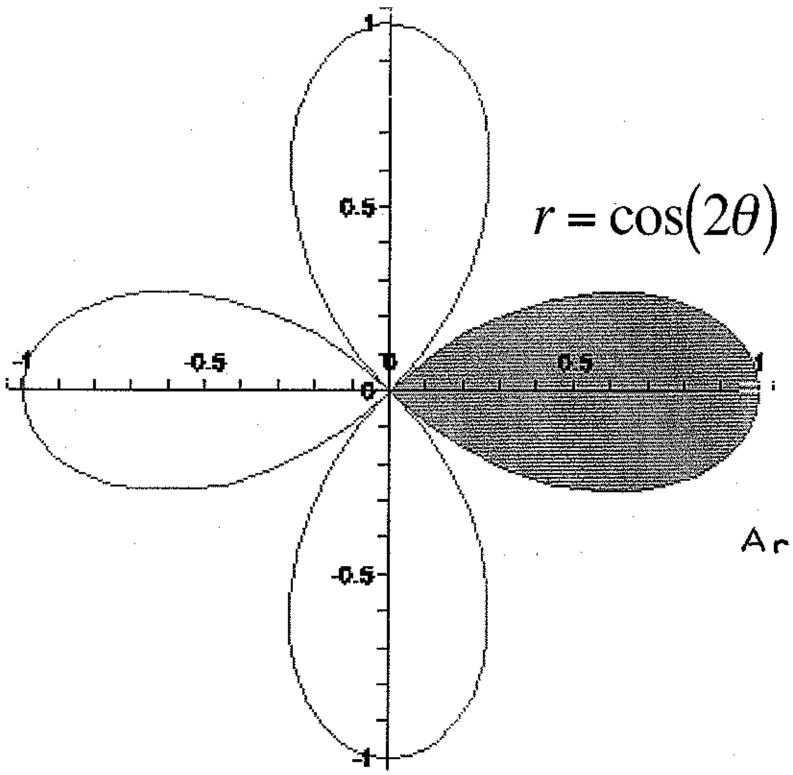
Find the **exact** area of each shaded region shown below. Show your work – no work = no credit.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(a) (5 points)



First determine the limits of integration by solving:

$$\cos(2\theta) = 0$$

$$2\theta = \pm \frac{\pi}{2}$$

$$\theta = \pm \frac{\pi}{4}.$$

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\pi/4} \cos^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + \cos(4\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{8}$$

Continued on the next page.

SOLUTIONS

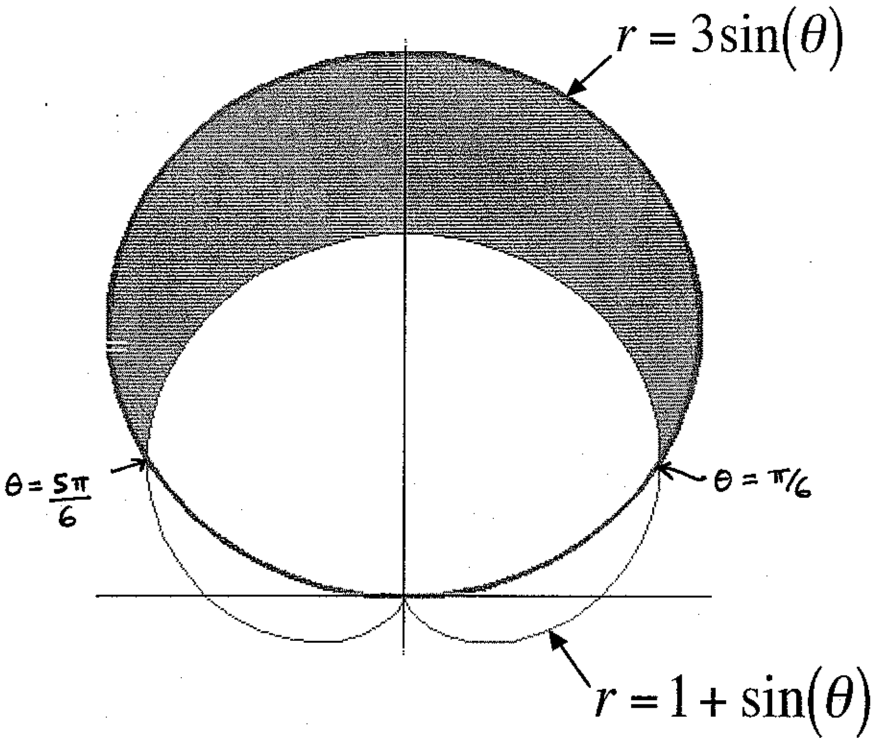
Find the area of each shaded region shown below. Show your work – no work = no credit.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(b) **(5 points)**



First, determine locations of intersection points.

$$1 + \sin(\theta) = 3\sin(\theta)$$

$$2\sin(\theta) = 1$$

$$\sin(\theta) = 1/2$$

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

These will be the limits of integration.

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2(\theta) - (1+\sin(\theta))^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\sin^2(\theta) - 1 - 2\sin(\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 - 4\cos(2\theta) - 1 - 2\sin(\theta)\, d\theta$$

$$= \frac{1}{2}\Big[3\theta - 2\sin(2\theta) + 2\cos(\theta)\Big]_{\pi/6}^{5\pi/6}$$

$$= \pi$$

5. 10 points total. SHOW YOUR WORK.

In this problem the function $f(x, y)$ will always refer to the function defined by the formula:

$$f(x,y) = x^2 + y^2 - x - y + 1.$$

(a) **(4 points)** Find the x and y coordinates of any critical points of $f(x, y)$. Record your results in the table below.

$f_x = 2x - 1$ $\qquad$ $f_y = 2y - 1$

(b) **(2 points)** Classify the critical points that you found in Part (a) as local maximums, local minimums or saddle points. Record your results in the table below.

$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0 \qquad D = f_{xx} f_{yy} - (f_{xy})^2 = 4 > 0$

$f_{xx} > 0.$

RECORD RESULTS FOR PARTS (A) AND (B) IN THIS TABLE:

x	y	Classification	$f(x, y)$
$1/2$	$1/2$	Local minimum	$1/2$

(c) **(4 points)** Find the global maximum and global minimum of $f(x, y)$ on the disk where $x^2 + y^2 \leq 1$. Show your work and **circle** your final answer(s).

Use Lagrange Multipliers. $\quad \nabla f = \langle 2x-1, 2y-1 \rangle$

$g(x,y) = x^2 + y^2 - 1 = 0. \quad \nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$: $\qquad 2x - 1 = 2\lambda x$ $\quad$ so $\quad x = y.$

$\qquad 2y - 1 = 2\lambda y$

Substitute $x = y$ into the constraint to get:

$2x^2 = 1 \qquad x = \pm 1/\sqrt{2} \qquad y = \pm 1/\sqrt{2}.$

x	y	$f(x,y)$	Comments
$1/\sqrt{2}$	$1/\sqrt{2}$	0.585786	
$-1/\sqrt{2}$	$-1/\sqrt{2}$	3.4142135	Global max.

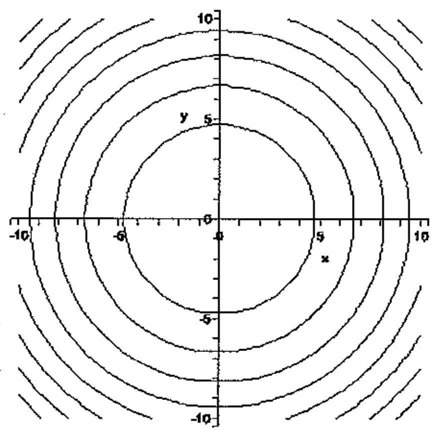
Global min of $f(x,y) = 1/2$

Global max of $f(x,y) = 3.4142135$

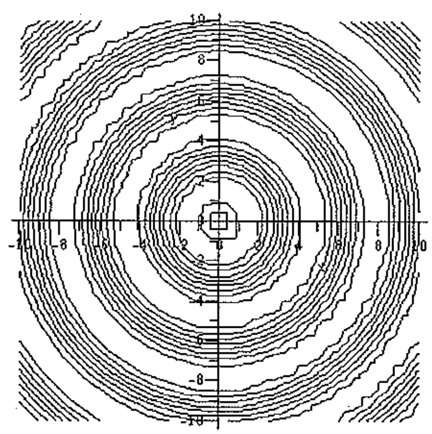
6. 10 points total. 2 POINTS FOR EACH CORRECT MATCH.

Match the contour plots with the functions given below. If you do not think that any of the contour plots does a good job of showing the level curves of a particular function, write the word **NONE** next to that function.

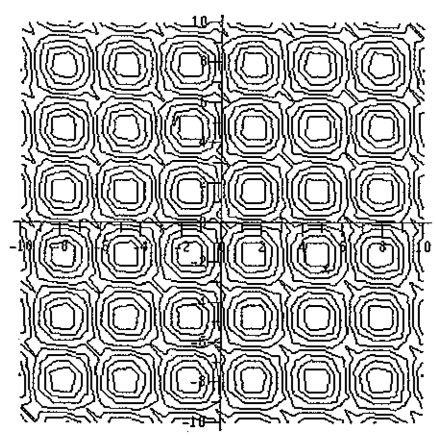
CONTOUR PLOT I



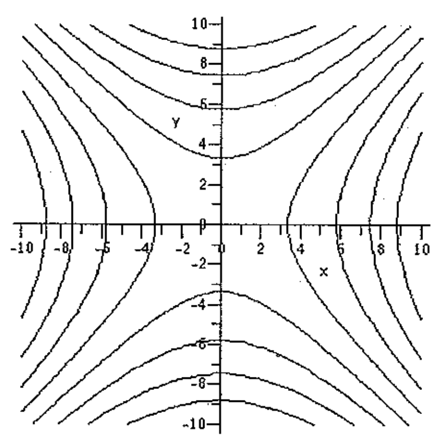
CONTOUR PLOT II



CONTOUR PLOT III



CONTOUR PLOT IV



(a) $f(x,y)=\sin(\sqrt{x^2+y^2})$ **CONTOUR PLOT =** II

(b) $f(x,y)=\dfrac{1}{\sin(x)}$ **CONTOUR PLOT =** NONE

(c) $f(x,y)=4-x^2-y^2$ **CONTOUR PLOT =** I

(d) $f(x,y)=x^2-y^2$ **CONTOUR PLOT =** IV

(e) $f(x,y)=\sin(x)\cdot\sin(y)$ **CONTOUR PLOT =** III

7. **8 Points. SHOW YOUR WORK.**

(a) **(4 points)** Find an equation for the plane that includes both the point (1, 3, 0) and the line

$$\langle x,y,z\rangle = \langle 2,7,1\rangle + t\cdot\langle -1,1,1\rangle.$$

Show all of your work and express your final answer in the form: $ax + by + cz = d$.

Let $\vec{v} = \langle 2-1, 7-3, 1-0\rangle = \langle 1, 4, 1\rangle$.

Normal vector $= \langle 1, 4, 1\rangle \times \langle -1, 1, 1\rangle$

$$= \begin{vmatrix} i & j & k \\ 1 & 4 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \langle 3, -2, 5\rangle$$

Equation of plane (many others possible):

$$3(x-1) - 2(y-3) + 5(z-0) = 0.$$

(b) **(4 points)** Let $f(x, y, z) = xyz$ and $g(x, y, z) = x^2 + y^2 - z$. Find a three-dimensional **unit vector** $\vec{u}$ that makes both of the directional derivatives equal zero:

$$D_{\vec{u}}f(1,1,1) = 0 \text{ and } D_{\vec{u}}g(1,1,1) = 0.$$

$\nabla f = \langle yz, xz, xy\rangle$ $\quad \nabla f(1,1,1) = \langle 1, 1, 1\rangle$.

$\nabla g = \langle 2x, 2y, -1\rangle$ $\quad \nabla g(1,1,1) = \langle 2, 2, -1\rangle$.

Want a vector $\vec{v}$ perpendicular to both.

$$\vec{v} = \langle 1, 1, 1\rangle \times \langle 2, 2, -1\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= \langle -3, 3, 0\rangle$$

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \left\langle -3/\sqrt{18}, 3/\sqrt{18}, 0\right\rangle$$

8. 7 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

Use Lagrange Multipliers to find the maximum and minimum values of the function:

$$f(x,y) = x - 2y - z$$

subject to the constraint:

$$x^2 + 3y^2 + z^2 = 1.$$

Show your work and record your answers in the space provided below.

$\nabla f = \langle 1, -2, -1 \rangle$ $\quad$ $g(x,y,z) = x^2 + 3y^2 + z^2 - 1 = 0$

$\nabla g = \langle 2x, 6y, 2z \rangle.$

$\nabla f = \lambda \nabla g$:

$$1 = 2\lambda x$$
$$-2 = 6\lambda y \quad \text{or} \quad -1 = 3\lambda y$$
$$-1 = 2\lambda z$$

λ is non-zero, so $x = \frac{1}{2\lambda}$, $y = \frac{-1}{3\lambda}$, $z = \frac{-1}{2\lambda}$.

This means $z = -x$ and $y = -\frac{2}{3}x$. Substituting these into $g = 0$ gives:

$$x^2 + 3\left(\frac{-2}{3}x\right)^2 + (-x)^2 = \frac{10}{3}x^2 = 1$$

so $x = \pm\sqrt{3/10}$.

x	y	z	$f(x,y,z)$
$\sqrt{3/10}$	$-2/3\sqrt{3/10}$	$-\sqrt{3/10}$	$\sqrt{10/3}$
$-\sqrt{3/10}$	$2/3\sqrt{3/10}$	$\sqrt{3/10}$	$-\sqrt{10/3}$

MAXIMUM VALUE OF $f(x, y)$: $\sqrt{10/3}$

MINIMUM VALUE OF $f(x, y)$: $-\sqrt{10/3}$

9. 7 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

Evaluate the surface integral:

$$\iint_S f(x,y,z)\,dS,$$

where $f(x, y, z) = 1$ and S is the surface parametrized by:

$$r(\theta,\varphi) = <(b + a\cdot\cos(\varphi))\cdot\cos(\theta), (b + a\cdot\cos(\varphi))\cdot\sin(\theta), a\cdot\sin(\varphi)>,$$

where a and b are positive constants with $0 < a < b$ and $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

$$\vec{r}_\theta = \langle -(b + a\cos(\varphi))\cdot\sin(\theta),\ (b + a\cos(\varphi))\cdot\cos(\theta),\ 0\rangle$$

$$\vec{r}_\varphi = \langle -a\sin(\varphi)\cos(\theta),\ -a\sin(\varphi)\sin(\theta),\ a\cos(\varphi)\rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(b+a\cos(\varphi))\sin(\theta) & (b+a\cos(\varphi))\cos(\theta) & 0 \\ -a\sin(\varphi)\cos(\theta) & -a\sin(\varphi)\sin\theta & a\cos(\varphi) \end{vmatrix}$$

$$= \langle (b+a\cos(\varphi))\cos(\theta)\,a\cos(\varphi),\ -(b+a\cos(\varphi))\sin(\theta)\,a\cos(\varphi),\ (b+a\cos(\varphi))\,a\sin(\varphi)\rangle$$

$$|\vec{r}_\theta \times \vec{r}_\varphi| = \sqrt{a^2\cdot(b + a\cdot\cos(\varphi))^2} = a(b + a\cos(\varphi))$$

$$\iint_S f(x,y,z)\,dS = \int_0^{2\pi}\int_0^{2\pi} ab + a^2\cos(\varphi)\ d\varphi\, d\theta$$

$$= 8\int_0^{2\pi}\Big[ab\varphi + a^2\sin(\varphi)\Big]_0^{2\pi} d\theta$$

$$= 8\int_0^{2\pi} 2\pi ab\ d\theta$$

$$= 4\pi^2 ab$$

SOLUTIONS

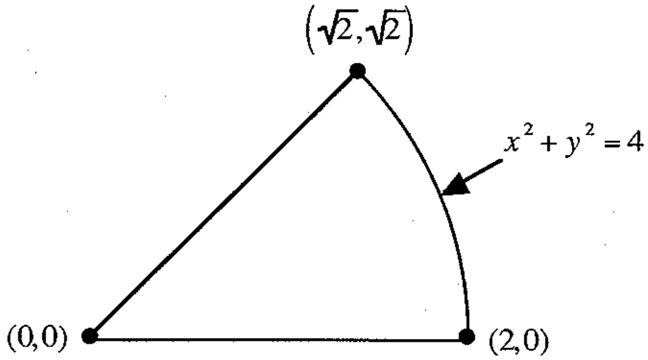
10. 10 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

(a) **(5 points)** Let C be the path shown in the diagram given below. Calculate the value of the path integral:

$$\int_C \left(y^2 - x^2 y\right) \cdot dy + xy^2 \cdot dx .$$

Use Green's Theorem with $P(x,y) = xy^2$ and $Q(x,y) = y^2 - x^2 y$. Then:



$$\int_C xy^2\,dx + (y^2 - x^2y)\,dy =$$

$$\int_0^{\pi/4} \int_0^2 \left(-2r^2\sin(\theta)\cos(\theta) - 2r^2\sin(\theta)\cos(\theta)\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left[-r^4 \sin(\theta)\cos(\theta) \right]_0^2 d\theta$$

$$= \int_0^{\pi/4} -16\sin(\theta)\cos(\theta)\,d\theta$$

$$= \left[-8\sin^2(\theta)\right]_0^{\pi/4} = -4.$$

(b) **(5 points)** Let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$, where a is a positive constant. Let $\vec{F}(x,y,z) = \langle x^3, y^3, z^3 \rangle$. Calculate the value of the surface integral:

$$\iint_S \vec{F} \bullet d\vec{S} .$$

Use the Divergence Theorem. $\nabla \bullet \vec{F} = 3(x^2 + y^2 + z^2)$.

$$\iint_S \vec{F} \bullet d\vec{S} = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} 3\rho^2 \cdot \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \frac{3a^5}{5} \int_0^{2\pi} \int_0^{\pi} \sin(\varphi)\,d\varphi\,d\theta$$

$$= \frac{3a^5}{5} \int_0^{2\pi} \left[-\cos(\varphi)\right]_0^{\pi} d\theta$$

$$= \frac{6a^5}{5} \int_0^{2\pi} d\theta$$

$$= \frac{12\pi a^5}{5}$$

11. 9 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

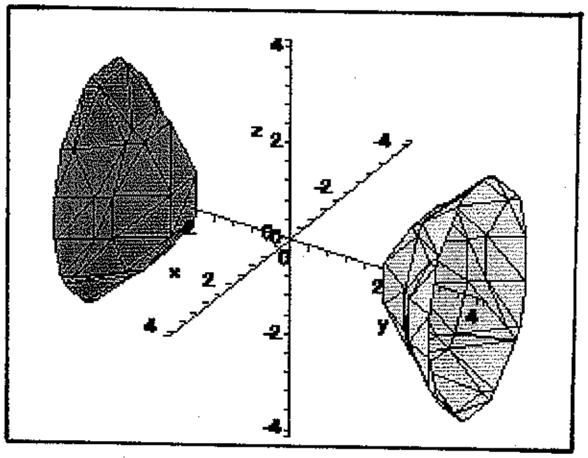
In this problem you will be interested in the surface defined by the equation:

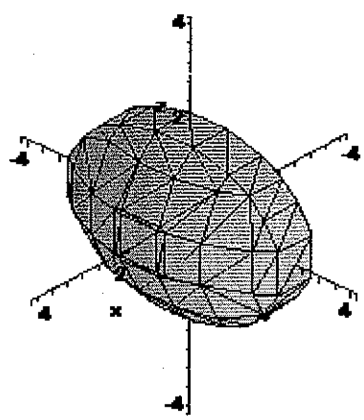
$$4x^2 - y^2 + 2z^2 + 4 = 0.$$

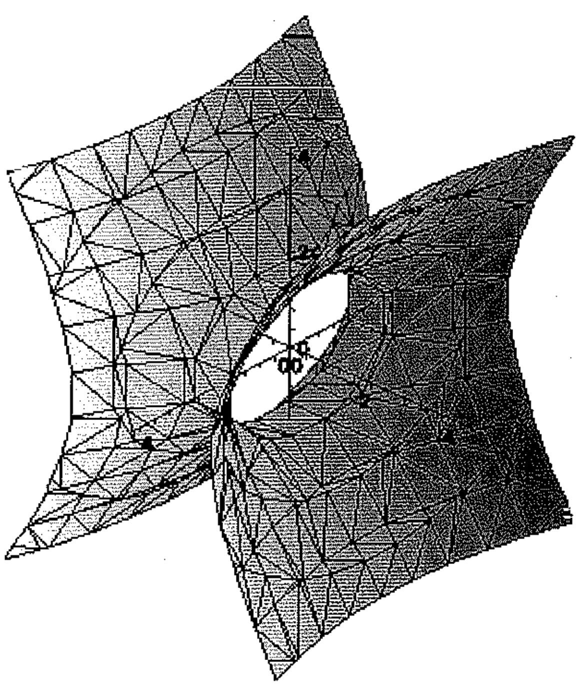
(a) **(2 points)** Classify the surface (i.e. is it a plane, ellipsoid, hyperboloid of two sheets, etc.?)

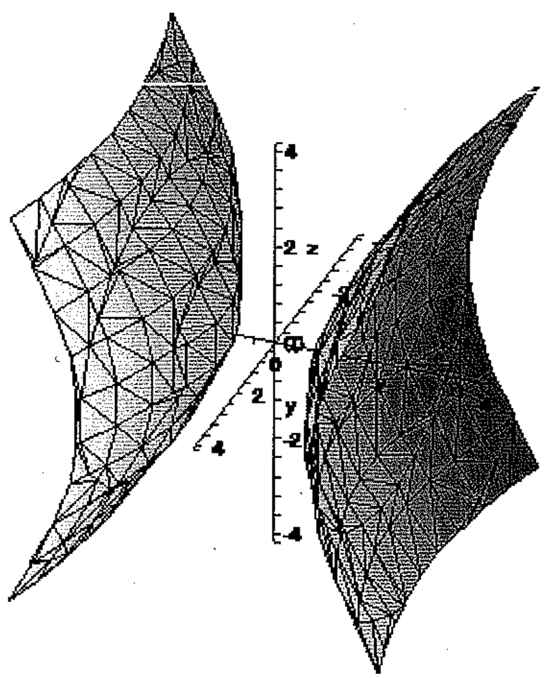
The surface is a hyperboloid of two sheets.

(b) **(2 points)** Which graph (of those shown below) does the best job of showing the graph of the surface defined by the equation given above? CIRCLE ONE (AND ONLY ONE) GRAPH.









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SHOW YOUR WORK. NO WORK = NO CREDIT.

In this problem you will be interested in the surface defined by the equation:

$$4x^2 - y^2 + 2z^2 + 4 = 0.$$

(c) **(5 points)** Find an equation for the tangent plane to the surface based at the point:

$$(x, y, z) = (1, 4, 2).$$

$$2z^2 = -4 - 4x^2 + y^2$$

$$z = \sqrt{-2 - 2x^2 + \tfrac{1}{2}y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(-2 - 2x^2 + \tfrac{1}{2}y^2\right)^{-1/2} \cdot (-4x)$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\left(-2 - 2x^2 + \tfrac{1}{2}y^2\right)^{-1/2} \cdot (y)$$

When $x = 1$ and $y = 4$: $\quad \dfrac{\partial z}{\partial x} = -1 \qquad \dfrac{\partial z}{\partial y} = 1$

Equation of tangent plane:

$$-(x - 1) + (y - 4) - (z - 2) = 0$$