SOLUTIONS

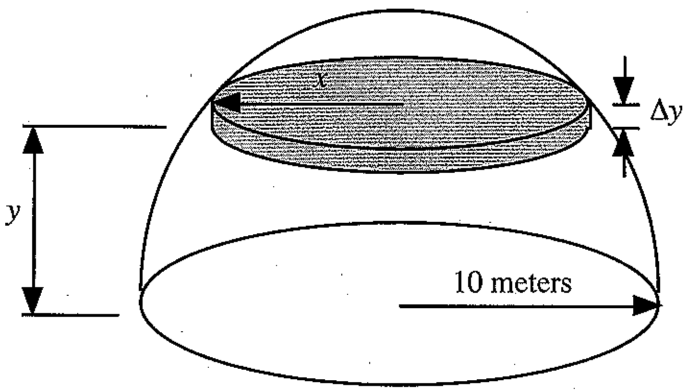
Quiz #4

For each question, be careful to indicate your final answer **and show how you obtained it**. Answers with no supporting work will get no credit.

You should not use your calculator to find antiderivatives on any of the problems on this quiz. You may use your calculator for arithmetic and to evaluate functions.

1. **(2 points)** Consider the shape shown below. This shape is the top half of a sphere with a radius of 10 meters. If you look at the shape "side on" then it looks like the top half of the circle described by the equation:

$$x^2 + y^2 = 100.$$



By slicing the shape into horizontal slices, set up an integral that gives the *volume* of the shape. There is no need for you to evaluate this integral – all that you need to do is to set the integral up.

The radius of the disk shown is x and its thickness is Δy. The volume is:

$$\text{Volume} = \pi(\text{radius})^2(\text{thickness})$$
$$= \pi x^2 \cdot \Delta y$$

Expressing this entirely in terms of y using $x^2 + y^2 = 100$ gives:

$$\text{Volume} = \pi \cdot (100 - y^2) \cdot \Delta y$$

Taking the limit as $\Delta y \to 0$, the total volume of the shape is given by:

$$\text{Total volume} \doteq \sum_i \pi(100 - y_i^2)\Delta y \xrightarrow[\Delta y \to 0]{\lim} \int_0^{10} \pi(100 - y^2)\,dy$$

$$\text{Total volume} = \int_0^{10} \pi(100 - y^2)\,dy$$

SOLUTIONS.

2. **(4 points)** The German airship *Hindenburg* is shown in the diagram given below. The *Hindenburg* is famous for crashing in a fiery blaze in Lakehurst, NJ, on May 6, 1937.

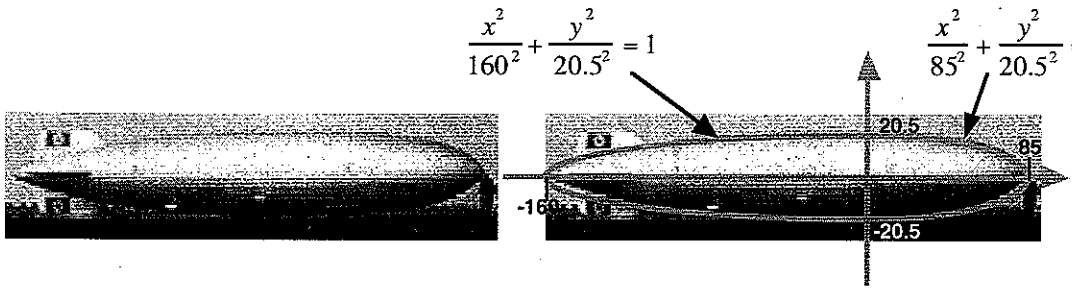


Figure 7: A mathematical model for the Hindenburg based on a pair of ellipses.

The airship had the shape of a volume of revolution. The front part of the airship was 85 meters long and the rear part of the airship was 160 meters long. Calculate the total volume of the *Hindenburg* in units of cubic meters.

The volume of the rear section of the airship is given by:

$$\text{Rear volume} = \int_{-160}^{0} \pi \cdot y^2 \cdot dx$$

$$= \int_{-160}^{0} \pi (20.5)^2 \cdot \left(1 - \frac{x^2}{160^2}\right) dx$$

$$= \pi \cdot (20.5)^2 \cdot \left[x - \frac{x^3}{(3)(160^2)} \right]_{-160}^{0}$$

$$= 140\,827.1267 \text{ m}^3$$

The volume of the forward section of the airship is given by:

$$\text{Forward volume} = \int_{0}^{85} \pi \cdot (20.5)^2 \cdot \left(1 - \frac{x^2}{85^2}\right) \cdot dx$$

$$= \pi \cdot (20.5)^2 \cdot \left[x - \frac{x^3}{(3)(85)^2} \right]_{0}^{85}$$

$$= 74\,814.41105 \text{ m}^3$$

Total volume of airship $= 215\,641.5378 \text{ m}^3$

SOLUTIONS.

3. Determine whether each of the following improper integrals *converges* or *diverges*. Indicate your answer by circling "CONVERGE" or "DIVERGE" next to each integral. If the integral converges, show why (e.g. calculate its value). If the integral diverges, show why.

Limited partial credit may be for correct, appropriate work (if shown) even if your final conclusion is incorrect. You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

(a) **(2 points)** $\int_{-3}^{0} \frac{y}{\sqrt{9-y^2}}\,dy$ **CONVERGE** **DIVERGE.**

$$\int_{-3}^{0} \frac{y}{\sqrt{9-y^2}}\,dy = \lim_{b \to -3^+} \int_{b}^{0} \frac{y}{\sqrt{9-y^2}}\,dy$$

Let $u = 9 - y^2$

$\frac{du}{dy} = -2y$

$dy = -\frac{du}{2y}$

$$= \lim_{b \to -3^+} \int_{9-b^2}^{9} -\frac{1}{2} u^{-1/2}\,du$$

$$= \lim_{b \to -3^+} \left[-u^{1/2} \right]_{9-b^2}^{9}$$

$$= \lim_{b \to -3^+} -3 + \sqrt{9-b^2}$$

$$= -3$$

$\therefore$ Improper integral converges to -3.

Continued over.

SOLUTIONS.

Determine whether each of the following improper integrals *converges* or *diverges*. Indicate your answer by circling "CONVERGE" or "DIVERGE" next to each integral. If the integral converges, show why (e.g. calculate its value). If the integral diverges, show why.

Limited partial credit may be for correct, appropriate work (if shown) even if your final conclusion is incorrect. You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

(b) **(2 points)** $\int_0^\infty \frac{x}{e^x}dx$ CONVERGE DIVERGE.

$$\int_0^\infty \frac{x}{e^x}\,dx = \int_0^\infty x \cdot e^{-x}\,dx$$

$$= \lim_{a \to \infty} \int_0^a x \cdot e^{-x}\,dx$$

use integration by parts

$u = x$ $\quad v' = e^{-x}$

$u' = 1$ $\quad v = -e^{-x}$

$$= \lim_{a \to \infty} \left[-xe^{-x}\right]_0^a + \int_0^a e^{-x}\,dx$$

$$= \lim_{a \to \infty} \left[-xe^{-x}\right]_0^a - \left[e^{-x}\right]_0^a$$

$$= 1$$

$\therefore$ Integral converges to 1.