SOLUTIONS.

Math 122 **Fall 2008**

Quiz #10

1. (a) **(4 points)** Find the Taylor Series for the function $f(x)=\dfrac{1}{1+x}$ around the point $a=3$. Express your final answer using sigma (Σ) notation, and clearly indicate your final answer.

We want to obtain an answer of the form:

$$\sum_{n=0}^{\infty} a_n (x-3)^n$$

so we will begin by manipulating $f(x)=\dfrac{1}{1+x}$ to create an $(x-3)$ term.

$$f(x) = \frac{1}{1+x} = \frac{1}{4+(x-3)} = \frac{1}{4}\cdot\frac{1}{1-\frac{-1}{4}(x-3)}$$

Using the summation formula for an infinite geometric series in reverse,

$$f(x) = \frac{1}{4}\cdot\frac{1}{1-\frac{-1}{4}(x-3)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-1}{4}(x-3)\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-3)^n$$

FINAL ANSWER:

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-3)^n$$

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(b) **(2 points)** Find the **radius** of convergence of the Taylor Series for the function $f(x) = \frac{1}{1+x}$ around the point $a = 3$.

$$a_n = \frac{(-1)^n}{4^{n+1}}(x-3)^n \qquad a_{n+1} = \frac{(-1)^{n+1}}{4^{n+2}}(x-3)^{n+1}$$

The ratio of consecutive terms is:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{4^{n+2}} \frac{(x-3)^{n+1}}{(x-3)^n} \frac{4^{n+1}}{(-1)^n} = \frac{-1}{4}(x-3).$$

Taking the limit of the absolute value of this ratio gives:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{-1}{4}(x-3)\right|$$

$$= \frac{1}{4}|x-3|.$$

Forcing the limit to be less than one gives the inequality:

$$\frac{1}{4}|x-3| < 1$$

$$|x-3| < 4.$$

So, the radius of convergence of the Taylor Series from (a) is 4.

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2. Consider the power series defined below.

$$1+\frac{2}{7^2}(x-3)+\frac{4}{7^3}(x-3)^2+\frac{8}{7^4}(x-3)^3+\frac{16}{7^5}(x-3)^4+\frac{32}{7^6}(x-3)^5+\ldots$$

(a) **(2 points)** Write out the sigma notation for this power series.

$$1+\sum_{n=1}^{\infty}\frac{2^n}{7^{n+1}}(x-3)^n$$

(b) **(2 points)** Determine the **interval of convergence** for this power series. Be sure to show your work. (Continue your work on the back of this sheet of paper if necessary, but write your final answer in the space provided below.)

For $n \geq 1$, $\quad a_n = \dfrac{2^n}{7^{n+1}}(x-3)^n \qquad a_{n+1} = \dfrac{2^{n+1}}{7^{n+2}}(x-3)^{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{7^{n+2}}\,\frac{(x-3)^{n+1}}{(x-3)^n}\,\frac{7^{n+1}}{2^n} = \frac{2}{7}(x-3)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2}{7}(x-3)\right| = \frac{2}{7}|x-3| < 1.$$

So the radius of convergence is: $r = 7/2$.

We also need to determine convergence at the end-points of the interval: $x = 3 - 7/2 = -1/2$ and $x = 3 + 7/2 = 13/2$.

FINAL ANSWER: The interval of convergence is:

$(-1/2,\ 13/2)$.

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$\underline{x = -1/2:}$ Plugging $x = -1/2$ into the power series gives:

$$1 + \sum_{n=1}^{\infty} \frac{2^n}{7^{n+1}} \left(-\tfrac{1}{2} - 3\right)^n = 1 + \sum_{n=1}^{\infty} \frac{2^n \cdot (-1)^n \cdot (7)^n}{7^{n+1} \cdot (2)^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{-1}{7}$$

This series does not converge (by the n^{th} term test), so $x = -1/2$ is not included in the infinite series.

$\underline{x = 13/2:}$ Plugging $x = 13/2$ into the power series gives:

$$1 + \sum_{n=1}^{\infty} \frac{2^n}{7^{n+1}} \left(\tfrac{13}{2} - 3\right)^n = 1 + \sum_{n=1}^{\infty} \frac{2^n}{7^{n+1}} \cdot \frac{7^n}{2^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{7}$$

This series does not converge (by the n^{th} term test), so $x = 13/2$ is not included in the infinite series.

So, the interval of convergence is the open interval from $x = -1/2$ to $x = 13/2$, or $(-1/2, 13/2)$.