Unit Test 3 Review Problems – Set A

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and may have to explain why your answers are reasonable and appropriate.

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1. Use the n^{th} term test for divergence to determine whether or not the following series diverge. If the n^{th} term test is inconclusive, use a different test to determine whether the series converges or not.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{3^n+4}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^4\cdot n!}{(n+4)!}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^2+n!}$ **(d)** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^{\frac{3}{2}}+\sqrt{n}}$

2. Use the Ratio Test to decide which series converge and which series diverge.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k!}.$ **(b)** $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{k^2+1}.$

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{2^k}{k!}$ **(d)** $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{2^k}.$

(e) $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{(2k)!}$ **(f)** $\displaystyle\sum_{k=1}^{\infty}\frac{2^{k+1}}{7^k}$

(g) $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^2}$

3. The ultimate point of this problem is to calculate the monthly payment for a mortgage. Throughout this problem, you should assume that this is an American mortgage – that means that the interest on the outstanding balance will be compounded monthly. Likewise, throughout the calculation (except for Part (e)) assume that the mortgage is a fixed-rate mortgage. That is, the interest rate will remain constant throughout the entire length of the mortgage.

(a) Suppose that a person borrowed \$200,000 at an annual interest rate of 7%. Let M represent the monthly mortgage payment (in dollars). Use this information to complete the table shown below.

Months since loan obtained	Amount still owed
1	
2	
3	
N	

(b) Suppose that the person has taken out a 30-year mortgage. Calculate the monthly payment, M, that the individual would have to send the the lending institution.

(c) Suppose that the person has taken out a 15-year mortgage. Calculate the monthly payment, M, that the individual would have to send the lending institution.

(d) If you have done the calculations in Parts (b) and (c) correctly, then you should have calculated that the monthly payment for the 15-year mortgage is considerably higher than the monthly payment for the 30-year mortgage. How much does the person end up paying the lender in each case?

(e) Suppose that instead of a fixed-rate interest mortgage, the person has obtained a variable rate interest mortgage. This means that the interest rate can vary throughout the term of the mortgage. Generally speaking, there will be periods when interests rates are low and periods when interest rates are high. Is it more advantageous to the person borrowing the money to have low interest to begin with and high interest at the end of the mortgage or vice versa? Briefly explain your reasoning.

4. Use the Integral test to determine whether the following infinite series converge or diverge. You might be able to see how to use other tests (besides the Integral test) to determine the convergence or divergence of each series. However, for practice, use the Integral test in each case. When using the Integral Test remember to:

(i) Verify that $f(x)$ obeys the conditions needed to use the Integral test, and,

(ii) Use correct limit notation when evaluating improper integrals.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{(2n-1)^2}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

(d) $\displaystyle\sum_{n=1}^{\infty}n\cdot e^{-n}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$

(f) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\cdot(\ln(n))^2}$

5. Use the Comparison test to determine whether the following infinite series converge or diverge. When making a comparison, make sure that you establish (in a step-by-step fashion using algebra) that the comparison is valid. You may use p-series for comparisons without having to establish that the p-series converges ($p > 1$) or diverges ($p \le 1$). If you want to compare to any other series, you will also need to establish the convergence or divergence of the series you are using for the comparison.

(a) $\displaystyle\sum_{n=7}^{\infty}\frac{1}{n-3}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{-n}}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4+4^n}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{2-\sin(n)}{n}$

(e) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{\ln(n)}$

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{2^n+1}$

6. The stimulant most commonly used in the United States is caffeine. This drug has been shown to enhance vigilance, increase alertness, to improve mood and to shorten reaction times. Traditionally consumed in beverages such as tea and coffee, caffeine can also be taken in the form of tablets such as the commercial product Vivarin®. Each Vivarin® tablet delivers 200 mg of caffeine. The half-life of caffeine in the human body is 6 hours for non-smokers.

When people take very large doses of caffeine (i.e. 500+ mg accumulate in their body) the drug can have some negative effects, however. Documented effects include anxiety attacks, tension, body tremors, insomnia, irregular heartbeat, uncontrollable sweating, elevated blood pressure, nausea, vomiting, diuresis (increased urination) and diarrhea.

Each Vivarin® tablet delivers 200 mg of caffeine. The amount of caffeine in a person's body is given by an exponential decay function. The half-life of caffeine in the body of a non-smoker is 6 hours. Let T represent the number of hours since a person took a Vivarin® tablet and $C(T)$ represent the amount (in mg) of caffeine in the person's body. Using some of the basic theory of pharmacokinetics, it is possible to determine that the formula for $C(T)$ is:

$$C(T) = 200\cdot(0.8908987181)^T.$$

(a) While studying all night a student takes Vivarin® in accordance with the manufacturer's recommendations – one tablet every four hours. Use the information described above to complete the table given below.

Number of hours that student has been taking Vivarin®	Number of Vivarin® tablets taken in addition to initial dose	Total amount of caffeine in the student's body is … (mg)
0	0	200
4	1	
8	2	
12	3	
$4\cdot N$	N	

(b) Write down a summation formula that will give the total amount of caffeine in the student's body after she has taken the initial dose and N additional doses of Vivarin®.

(c) When a person accumulates a large amount of caffeine in their body (500+ mg) they start to experience adverse reactions. After how many hours of taking Vivarin® will the student start to experience adverse reactions?

(d) If the student decides to keep taking one Vivarin® tablet every four hours indefinitely, how much caffeine (in mg) will ultimately build up in her body?

7. In this problem you will consider the infinite series: $\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n}$.

(a) Use a convergence test to show that this is a convergent series.

(b) Use the methods that you used when studying the Technique of Partial Fractions to verify the algebraic identity:

$$\frac{2}{n^2 + 2n} = \frac{1}{n} - \frac{1}{n+2}.$$

(c) Let S_N represent the N^{th} partial sum of the series. Find a formula for S_N that is valid when $N > 3$.

(d) Find the sum of the infinite series.

8. Each of the series listed below is an alternating series. For each series:

(i) Determine if the series is absolutely convergent, conditionally convergent or divergent.

(ii) If the series is convergent, determine a value of N that will allow the sum of the series to be approximated by a partial sum, S_N, with an error at most equal to the error given.

(iii) If the series is convergent, evaluate S_N with the value of N calculated in (ii).

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2^n}{n^2}$, error < 0.001.

(b) $\sum_{n=1}^{\infty} (-1)^{n} \cdot \frac{2^n}{3^n}$, error < 0.01.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$, error < 0.1.

(d) $\sum_{n=1}^{\infty} (-1)^{n} \cdot \left(1 + \frac{1}{n}\right)$, error < 0.001.

9. Use convergence and divergence tests to determine whether each of the following series converges or diverges. You may use any test for any series but check to make sure that the specific conditions that any test you use are met before using it.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n-1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3n-1} - \frac{1}{3n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1+n}{4n}\right)^n$

Hint for Part (d): $\displaystyle\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$

10. Suppose that the terms, a_k, of the infinite series $\displaystyle\sum_{k=1}^{\infty} a_k$ are all positive numbers and that the infinite series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. Define a positive number b_n by the formula:

$$b_n = \frac{1}{n} \cdot \sum_{k=1}^{n} a_k = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

(a) Show that $\displaystyle\lim_{n \to \infty} b_n = 0$.

(b) What does the result of Part (a) tell you about the convergence or divergence of the infinite series $\displaystyle\sum_{n=1}^{\infty} b_n$?

(c) Does the infinite series $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge? Justify your answer by using an appropriate convergence or divergence test.