Unit Test 2 Review Problems – Set B

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and may have to explain why your answers are reasonable and appropriate.

1. Consider the region R that consists of the part of the first quadrant bounded by the curves:

$$y = \sin(x) \qquad y = x \qquad x = 0 \qquad x = 1.$$

(a) A solid S is formed by revolving the region R around the x-axis. Write down an integral that will give the exact volume of the solid S.

(b) A solid D is formed by revolving the region R around the y-axis. Write down an integral that will give the exact volume of the solid D.

(c) A solid F has the region R as its base. The cross-sections of F perpendicular to the x-axis are squares. Write down an integral that gives the exact volume of the solid F.

(d) A solid G has the region R as its base. The cross-sections of G perpendicular to the y-axis are semi-circles (half circles). Write down an integral that gives the exact volume of the solid G.

(e) Find an expression that will give the exact perimeter of the region R. You do not have to find the exact *value* of the perimeter. Your expression may involve integrals. If your expression does include any integrals, you **do not** have to evaluate them.

2. A water treatment pond is in the shape of a solid of revolution. With all dimensions given in meters, the water treatment pond takes the shape of the following region R revolved around the y-axis.

The region R is the region bounded by the following curves:

$$y = 0 \qquad y = \tfrac{3}{69}x^2 - \tfrac{100}{23} \qquad y = 3 \text{ and } \qquad x = 0.$$

(a) Set up and evaluate an integral to find the total volume of the pond.

(b) A rectangular plate of metal with a width of 1 meter is located in the exact center of the pond. The plate is tall enough so that its top is above the level of the water when the pond is full of water. Suppose that the pond is full of pure water (density 1000 kg/m^3). Calculate the hydrostatic force exerted on each side of the metal plate.

(c) Now suppose that the pond is completely full of polluted water that has a uniform density of 1100 kilograms per cubic meter. Find the exact amount of work that must be done to pump all of the polluted water out of the pond.

3. Carbon dioxide is a gas that is necessary for plants to grow. Some horticulturalists are experimenting with pumping carbon dioxide into a greenhouse to increase plant growth. In a greenhouse of volume 350 m^3, air containing 10% carbon dioxide is introduced at a rate of 0.005 m^3 per minute. (This means that 10% of the volume of the incoming air is carbon dioxide.) The carbon dioxide mixes immediately with the rest of the air and the mixture leaves the room at the same rate it enters.

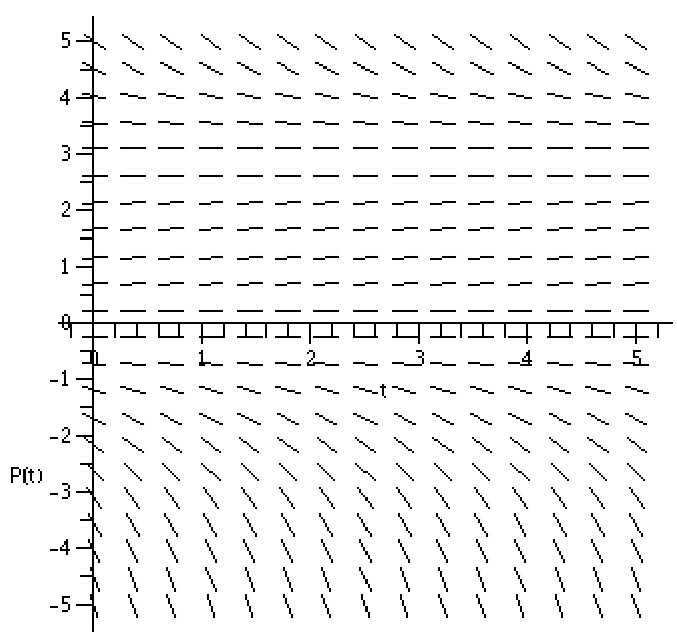
(a) Write a differential equation for $P(t)$, the percentage of the air in the greenhouse that consists of carbon dioxide at time t, in minutes.

(b) Suppose that initially, 0.05% of the air in the greenhouse is carbon dioxide. Solve to find a formula for $P(t)$.

(c) According to the National Institute of Occupational Safety and Health (NIOSH), carbon dioxide levels of 4% or higher are immediately dangerous to life and health. How many minutes does it take for the percentage of carbon dioxide in the greenhouse to reach this level?

(d) The percentage of carbon dioxide in fresh air is normally about 0.05%. Plant growth can be increased by up to 50% if the carbon dioxide content is raised to 0.1%. How many minutes does it take for the percentage of carbon dioxide in the greenhouse to reach this level?

4. Consider the following differential equation: $\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{3}\right)$. The slope field of this differential equation is shown below.



(a) Find and classify all equilibrium solutions of the differential equation.

(b) Consider the solution of the differential equation that passes through the point (0, –1). Use the slope field provided above to sketch the graph of this solution.

(c) Let $P(t)$ refer to the solution of the differential equation that passes through the point (0, –1). Using a value of $\Delta t = 0.5$, use Euler's method to estimate the numerical value of $P(2)$.

(d) Is the estimate that you calculated in Part (c) an overestimate or an underestimate of $P(1)$? Briefly explain (in a sentence or two) how you know.

5. Endostatin is an experimental drug that might be useful in the fight against cancer. When it is used, a patient begins with no endostatin in their body. The drug is then supplied to the patient by a pump at the steady rate of 25 mg per hour. The human body eliminates endostatin at a rate that is proportional to the amount of endostatin in the body. The constant of proportionality is 2.7.

The function $y = E(t)$ will represent the amount of endostatin in a patient's body (measured in mg) t hours after the treatment has begun.

(a) Use the information given above to create a differential equation and an initial condition that describe the rate of change of the amount of endostatin in a patient's body.

(b) Find and classify any equilibrium solutions that the differential equation from Part (a) may have. In a sentence or two explain the practical significance of each equilibrium solution in terms of the level of endostatin in a patient's blood.

(c) Find a formula for $E(t)$. Your final answer should contain no unspecified constants.

(d) Endostatin is effective when at least 9 mg of the drug are present in a patient's body. How many hours of treatment does it take for the amount of endostatin in a patient's body to reach effective levels?

6. In this problem, the differential equation will always be:

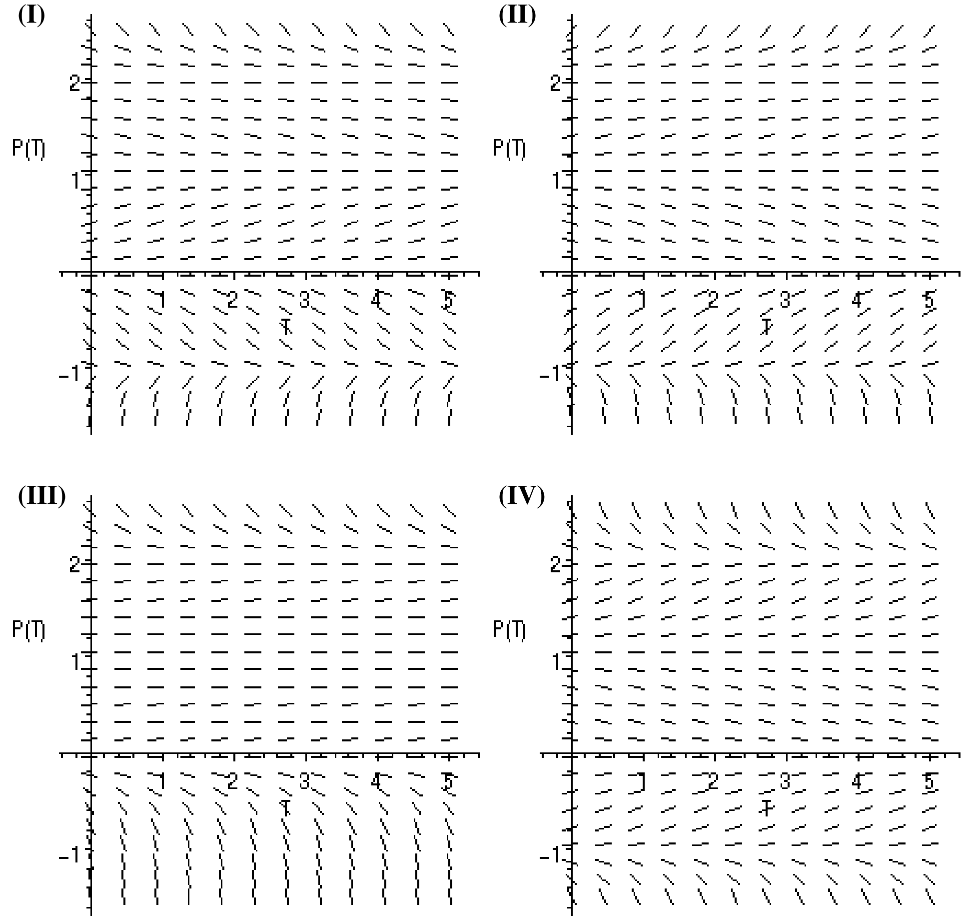
$$P'(T) = 0.25 \cdot P(T) \cdot \left[P(T)+1\right] \cdot \left[P(T)-1\right] \cdot \left[2-P(T)\right]^2.$$

(a) Locate all of the equilibrium solutions of this differential equation.

(b) Use the differential equation to complete the table given below.

T	$P(T)$	$P'(T)$
0	2.5	
1	1.5	
2	0.5	
3	–0.5	
4	–1.5	

(c) Four slope fields are shown on the next page (I-IV). Determine which of these slope fields corresponds to the differential equation given in this problem.



(d) Classify each of the equilibrium solutions that you found in Part (a) as stable, unstable or semi-stable.

7. Solve each of the following initial value problem. Your answers should not contain any unspecified constants.

(a) $y'' - 3y' - 4y = 0$ $\quad y(0) = 1$ $\quad y'(0) = 0.$

(b) $y'' + 6y' + 10y = 0$ $\quad y(0) = 0$ $\quad y'(0) = 2.$

(c) $y' + y^2 \cdot \sin(x) = 0$ $\quad y(0) = 1.$

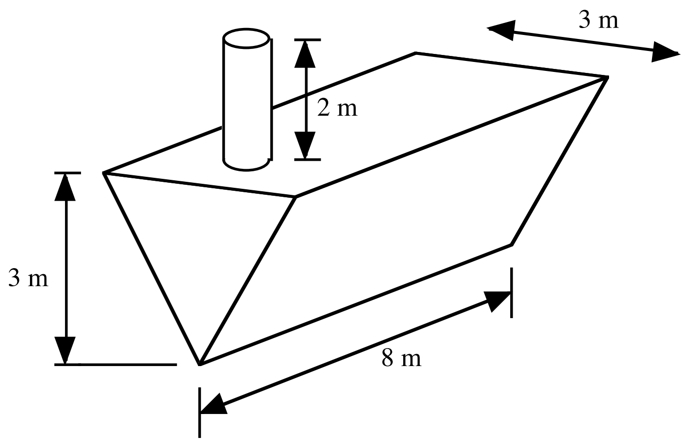
(d) $y'' + 5y' + 6y = 0$ $\quad y(0) = 1$ $\quad y'(0) = 0.$

8. A water tank has the shape of a triangle lying on its side as shown in the diagram on the next page. The tank is filled with pure water (density = 1000 kg/m^3). The tank is buried in the ground and there is a 2-meter tall pipe that leads from the tank to the surface. There is no water in this vertical pipe to start with.

(a) Suppose that the tank is completely full of water. Calculate the hydrostatic force exerted on each of the triangular ends of the tank.

(b) Suppose that the tank is now emptied completely of water. Calculate the total work that is needed to pump all of the water out of the tank.

(c) The pump used to remove water from the tank breaks down after doing 4.7×10^5 J of work. What is the depth of the water remaining in the tank?



9. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction of population who have heard the rumor and the fraction of the population who have not heard it. Let $f(t)$ represent the fraction of the population of a small town who have heard rumor that unflattering pictures of a local politician have appeared on the Internet, where t is the number of hours after 8am.

(a) Write down a differential equation to describe the spread of the rumor. Your differential equation should have exactly one unspecified constant in it.

(b) Use the technique of Separation of Variables to solve the differential equation from Part (a). The solution of your differential equation should have exactly two unspecified constants in it.

(c) The small town in question has exactly 1000 residents. At 8am, 80 people had heard the rumor. By noon, half the town had heard it. At what time of the day will 90% of the people in the small town have heard the rumor?

10. Find the total area of the region bounded by each of the following sets of curves.

(a) $y = \cos(x)$, $y = \sin(2x)$, $x = 0$ and $x = \pi/2$.

(b) $y = x^2 - x$ and $y = x^3 - 4x^2 + 3x$.

(c) $x = 2y^2$ and $x = 4 + y^2$.