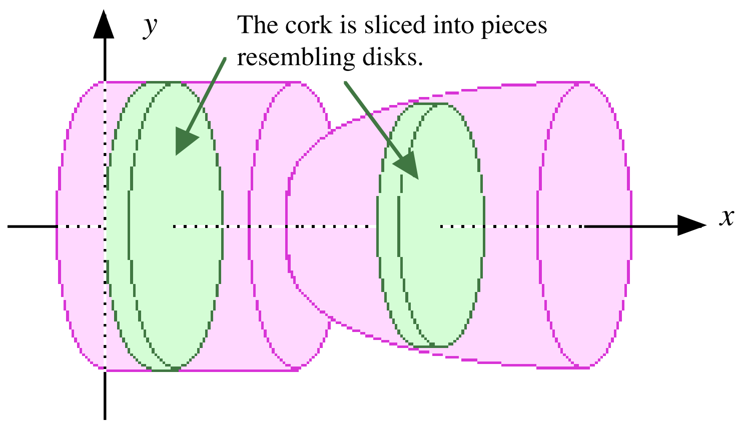
Math 122, Fall 2008. Answers to Unit Test 2 Review Problems – Set A.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) A conventional cork is cylindrical with a radius of about 0.9cm and a height of about 4.4cm. The volume of such a cork in cubic centimeters is:

$$Volume = \pi \cdot r^2 \cdot h \approx 11.197 \text{ cubic centimeters.}$$

1.(b) The diagram showing the 3-d shape, together with the disk-shaped slices is shown below.



1.(c) The portion of the champagne cork nearest the *y*-axis is a cylinder with a radius of 1.5cm and a height of 2cm. The volume of this portion of the cork (in cubic centimeters) is given by:

$$Volume = \pi \cdot r^2 \cdot h \approx 14.137 \text{ cubic centimeters.}$$

1.(d) The limits of integration are the *x*-coordinate of the left side of the area from Figure 7 that is revolved around the *x*-axis to create the cork shape (for the lower limit of integration) and the *x*-coordinate of the right side of this area (for the upper limit of integration). The limits of integration will be:

- Lower limit of integration = 2.
- Upper limit of integration = 5.

The expression that will be integrated will be the volume formula for the disks that the cork is sliced into. This is given by:

$$Volume\ of\ Disk = \pi \cdot y^2 \cdot dx = \pi \cdot \left[\sqrt{\tfrac{5}{12} \cdot x + \tfrac{2}{12}}\right]^2 \cdot dx = \pi \cdot \left[\tfrac{5}{12} \cdot x + \tfrac{2}{12}\right] \cdot dx.$$

The integral that gives the volume of the curved portion of the cork is:

$$Volume = \int_2^5 \pi \cdot \left[\tfrac{5}{12} \cdot x + \tfrac{2}{12}\right] \cdot dx.$$

1.(e) Evaluating the integral from Part (d) of this problem will give the volume of the curved portion of the cork in units of cubic centimeters.

$$\textit{Volume} = \int_2^5 \pi \cdot \left[\tfrac{5}{12} \cdot x + \tfrac{2}{12}\right] \cdot dx = \pi \cdot \left[\tfrac{5}{24} \cdot x^2 + \tfrac{2}{12} \cdot x\right]_2^5 \approx 15.315 \text{ cubic centimeters.}$$

1.(f) The total volume of the champagne cork is about (14.137 + 15.315) = 29.452 cubic centimeters. This is about three times the volume of a conventional cork.

2.(a) Slicing perpendicularly to the long side of the lake gives:

$$\text{Volume} = \int_0^{150} \tfrac{1}{2}(3)(0.2)\,dx.$$

2.(b) Slicing horizontally gives:

$$\text{Volume} = \int_0^{0.2} \left(\tfrac{3}{0.2} y\right)(150)\,dy.$$

2.(c) In both cases the integral works out to be 45 cubic kilometers of water.

3.(a) Volume = $\frac{\pi}{5}$.

3.(b) Volume = $\frac{256\pi}{15}$.

3.(c) Volume = $\frac{\pi\left(e^2 - e^{-2}\right)}{2}$.

3.(d) Volume = $\frac{\pi}{2}$.

3.(e) Volume = $\frac{2\pi}{15}$.

4.(a) Total mass = $\int_{-1}^{0} (1+x)^2 dx + \int_0^1 1 - x^2 dx = \frac{1}{3} + \frac{2}{3} = 1$.

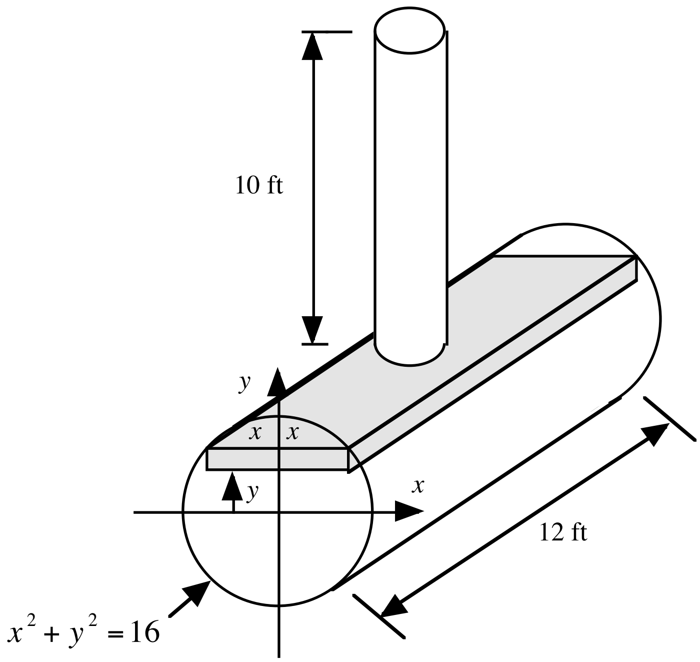
4.(b) I would expect the x-coordinate of the center of mass to be to the right of the x-axis (that is, to have a positive x value). This is because the center of mass (or balance point) is usually closer to the end of the object where the density (and depending on the shape, also usually the mass) is the greatest. As the density function for the triangular object is $\delta(x) = 1 + x$, the density is the highest near $x = +1$, so I would expect the center of mass to be closer to $x = +1$ than to $x = -1$ at the other end of the object.

4.(c) The integral that will give the denominator for the center of mass is the integral from 4.(a) which came out to be 1. The integral that will give the numerator for the center of mass is the following integral (which consists of exactly the same integrals as 4.(a) but each with an additional factor of x):

$$\int_{-1}^{0} x \cdot (1+x)^2 dx + \int_0^1 x - x^3 dx = \tfrac{-1}{12} + \tfrac{1}{4} = \tfrac{1}{6}.$$

So the x-coordinate of the center of mass is: $x = \frac{1}{6}$. (A positive number, which confirms the intuition expressed in 4.(b).)

5. We will slice the gasoline in the tank into horizontal slices. This is because all of the gasoline molecules that lie in a thin horizontal slice will travel approximately the same distance as they are moved to the surface and out of the gasoline tank. Such a slice is shown in gray on the diagram below.



The distance that the gray slice moves is $14 - y$ feet. The force exerted on it by gravity will be equal to the density of the rusty gasoline times the volume of the gray slice. The density of the gasoline is $42 - 0.1y$ pounds per cubic foot and the volume of the gray slice is $2x$ times 12 times dy. The width of $2x$ can be converted to a formula that involves y because $x^2 + y^2 = 16$, so $2x = 2\sqrt{16 - y^2}$. Multiplying all of these factors together gives the work needed to pump the gasoline in the gray slice out of the tank:

$$\text{Work} = (14 - y)\cdot(42 - 0.1y)\cdot(2)\cdot\sqrt{16 - y^2}\cdot(12)\cdot dy \text{ ft-lbs.}$$

The total amount of work to pump all of the gasoline out of the tank will be given by the following integral:

$$\int_{-4}^{4}(14 - y)\cdot(42 - 0.1y)\cdot(2)\cdot\sqrt{16 - y^2}\cdot(12)\cdot dy \approx 354914.52 \text{ ft-lbs.}$$

Note that this integral was evaluated using the numerical integration capabilities of a graphing calculator (as permitted in the statement of the problem).

6.(a) Mass =

$$\int_0^H k\cdot(H - x)\cdot\pi\cdot\left(\tfrac{-R}{H}x + R\right)^2\cdot dx = \pi\cdot k\cdot R^2\cdot\int_0^H\left(H - 3x + \tfrac{3}{H}x^2 - \tfrac{1}{H^2}x^3\right)dx = \frac{\pi\cdot k\cdot R^2\cdot H^2}{4}.$$

6.(b) The denominator in the fraction that will give the location of the center of mass is the answer to 6.(a). The integral that will give the numerator for the center of mass is given by:

$$\pi \cdot k \cdot R^2 \cdot \int_0^H \left(Hx - 3x^2 + \tfrac{3}{H} x^3 - \tfrac{1}{H^2} x^4 \right) dx = \frac{\pi \cdot k \cdot R^2 \cdot H^3}{20}$$

The *x*-coordinate (*x* being the vertical distance from the ground) of the center of mass is given by:

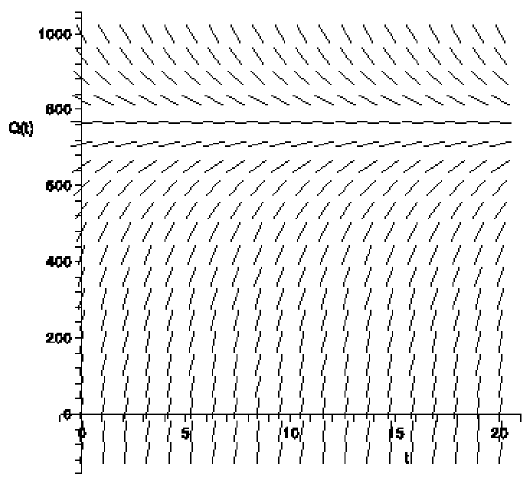
$$\bar{x} = \frac{\pi \cdot k \cdot R^2 \cdot H^3}{20} \cdot \frac{4}{\pi \cdot k \cdot R^2 \cdot H^2} = \frac{H}{5}.$$

6.(c) The total amount of work done by gravity is given by the integral:

$$\text{Work} = \int_0^H (H - x) \cdot k \cdot (H - x) \cdot (9.8) \cdot \pi \cdot \left(\tfrac{-R}{H} x + R \right)^2 \cdot dx \text{ J.}$$

7.(a) The differential equation is: $\dfrac{dQ}{dt} = 300 - 0.4 \cdot Q.$

7.(b) There is exactly one equilibrium solution of this differential equation. It is: $Q = 750$. If you sketch a slope field (see below) then you will be able to see that it is a stable equilibrium.



7.(c) The calculation to find the formula for $Q = Q(t)$ is carried out as follows:

$$\frac{dQ}{dt} = -0.4 \cdot (Q - 750)$$

$$\int \frac{1}{Q - 750} \cdot dQ = \int -0.4 \cdot dt$$

$$\ln(|Q - 750|) = -0.4 \cdot t + C$$

$$Q - 750 = A \cdot e^{-0.4 \cdot t} \qquad \text{where} \qquad A = \pm e^{C}.$$

Using $Q(0) = Q_0$ and rearranging to make Q the subject of the equation we get:

$$Q = Q(t) = 750 + (Q_0 - 750) \cdot e^{-0.4 \cdot t}.$$

7.(d) As $t \to \infty$, the limit that $Q(t)$ approaches is 750. This means that in the long run, about 750 ml of sports drink will accumulate in the athlete's body.

7.(e) Yes it is possible so long as $0 < Q_0 < 375$. The time required to double the sport fluid is given by solving the equation:

$$2 \cdot Q_0 = 750 + (Q_0 - 750) \cdot e^{-0.4 \cdot t}.$$

for t. The solution of this equation is: $$t = \frac{-1}{0.4} \cdot \ln\left(\frac{2 \cdot Q_0 - 750}{Q_0 - 750}\right).$$

8.(a) The table showing the work involved in carrying out Euler's method is given below. According to the figures given in this table, $f(1) \approx 3.99062$.

Current x	Current f(x)	Derivative	Rise	New f(x)
0	1	1	0.25	1.25
0.25	1.25	1.625	0.40625	1.65625
0.50	1.65625	2.99316	0.74829	2.40454
0.75	2.40454	6.344317	1.58607	3.99062

8.(b) The approximation $f(1) \approx 3.99062$ is an underestimate of the true value of $f(1)$. This is because the increasing derivatives in the table from Part (a) show us that the function $f(x)$ is concave up and Euler's method gives an underestimate when the function approximated is concave up.

8.(c) No, there are no equilibrium solutions. This is because there is only one point in the xy-plane that can solve the equation $x^2 + y^2 = 0$, which is $x = y = 0$. There is no curve in the xy-plane along which the derivative is equal to zero.

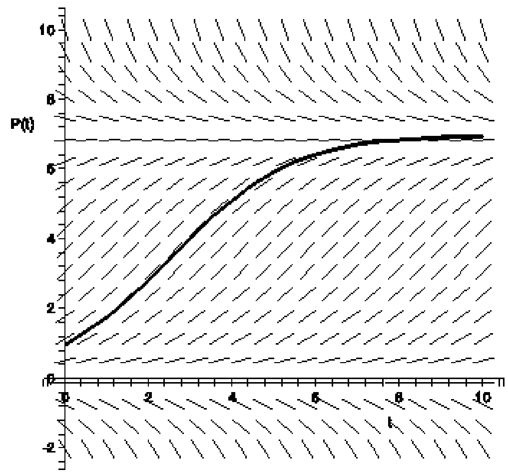
9.(a) The equilibrium solutions are the horizontal lines located at $P = 0$ and $P = 7$.

9.(b) The slope field is shown on the next page. The appearance of the slope field around the horizontal lines $P = 0$ and $P = 7$ shows that $P = 0$ is an unstable equilibrium and $P = 7$ is a stable equilibrium.

9.(c) The graph showing world shrimp production (starting with initial value $P(0) = 1$) is shown on the slope field given on the next page. The significance of the equilibrium solution at $P = 0$ is that is there was never any world shrimp production, then there would never be any world shrimp production; the graph of $P(t)$ would be a horizontal line at $P = 0$. The significance of the equilibrium solution at $P = 7$ is that it suggests that the maximum sustainable level of world shrimp production will be about $P = 7$, or 700,000 metric tons of shrimp per year.

9.(d) The work for Euler's method is summarized in the table on the next page. From the table, we can see that $P(2) \approx 2.6323$.

9.(e) The increasing derivatives in the table on the following page show that for the region of the P-t plane that we are interested in, the function $P(t)$ is concave up. When approximating a concave up function, Euler's method gives an underestimate of the true function value.



Current t	Current P	Derivative	Rise	New P
0	1	0.6	0.3	1.3
0.5	1.3	0.741	0.3705	1.6705
1.0	1.6705	0.89029	0.4451	2.1156
1.5	2.1156	1.0333	0.5166	2.6323

9.(f) The work to solve for $P(t)$ is shown below.

$$\frac{dP}{dt} = -0.1 \cdot P \cdot (P-7)$$

$$\int \frac{1}{P \cdot (P-7)} \cdot dP = \int -0.1 \cdot dt$$

$$\int \left(\frac{\frac{-1}{7}}{P} + \frac{\frac{1}{7}}{P-7} \right) \cdot dP = \int -0.1 \cdot dt$$

$$\tfrac{-1}{7} \cdot \ln(|P|) + \tfrac{1}{7} \cdot \ln(|P-7|) = -0.1 \cdot t + C$$

$$\ln\left(\left| \frac{P-7}{P} \right| \right) = -0.7 \cdot t + C$$

$$\frac{P-7}{P} = A \cdot e^{-0.7 \cdot t} \qquad \text{where } A = \pm e^{C}.$$

$$P - 7 = A \cdot e^{-0.7 \cdot t} \cdot P$$

$$P - A \cdot e^{-0.7 \cdot t} \cdot P = 7$$

$$P = \frac{7}{1 - A \cdot e^{-0.7 \cdot t}}.$$

Using the initial condition $P(0) = 1$ to solve for A gives $A = -6$, so that the formula for $P(t)$ is:

$$P = \frac{7}{1 + 6 \cdot e^{-0.7 \cdot t}}.$$

9.(g) Plugging $t = 2$ into the formula from Part (f) gives $P(2) = 2.823$. This is larger than the estimate produced by Euler's method confirming that Euler's method gives underestimates of true function values when used to approximate functions that are concave up.

10.(a) $y(t) = C_1 \cdot e^{4t} + C_2 \cdot e^{-t}$.

10.(b) $A = 40/52$ and $B = 8/52$.

10.(c) $y(t) = \frac{-96}{260} \cdot e^{4t} + \frac{156}{260} \cdot e^{-t} + \frac{40}{52} \cdot e^{t} \cdot \cos(2t) + \frac{8}{52} \cdot e^{t} \cdot \sin(2t)$