Unit Test 2 Review Problems – Set A

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and may have to explain why your answers are reasonable and appropriate.

1. Corks for wine bottles are usually from strips of bark from the cork tree (*Phellodendron amurense*). A cork from a typical bottle is a roughly cylindrical shape whose volume can be determined with the formula:

$$V = \pi \cdot r^2 \cdot h$$

where r is the radius of the cork and h is the height of the cork.

Simple cylindrical corks are most suitable for "still" wines. That is, wines that do not have any carbonation due to dissolved CO_2 present in the wine.

Figure 6: Cork for a bottle of Champagne.

Corks for sparkling wines (the best known being the wines from the Champagne region of France) are designed quite differently (see photograph[1]). This is because the contents of a bottle of sparkling wine are under very high pressure (often five times regular atmospheric pressure or higher). The cork for a bottle of sparkling wine has to both seal the bottle (the function of any cork) and to resist the strong forces exerted on it by the pressurized contents of the wine bottle.

In this problem you will calculate the volume of a champagne cork and the volume of a conventional cork (for the sake of comparison).

(a) A conventional cork from a bottle of still wine typically measures about 4.4cm in length with a radius of about 0.9 cm. Calculate the volume of a conventional cork in cubic centimeters.

(b) Figure 7 shows a shaded area, which when revolved around the x-axis creates a reasonable approximation for the shape of a cork from a bottle of Perrier-Jouet Brut Imperial Champagne, vintage 1994. Sketch a three dimensional picture of the shape that will be formed by revolving the shaded area from Figure 7 about the x-axis and indicate how you can "slice" this shape up into simpler pieces.

(c) The portion of the cork that is nearest to the y-axis can be approximated by a cylinder. Calculate the volume of this portion of the champagne cork in units of cubic centimeters.

(d) The outline of the remaining portion of the champagne cork is described by the equation:

$$y = \sqrt{\tfrac{5}{12} \cdot x + \tfrac{2}{12}}$$

valid from $x = 2$ to $x = 5$. Set up an integral whose numerical value will equal the volume of this portion of the champagne cork in units of cubic centimeters.

[1] Image source: http://www.cyberbacchus.com/

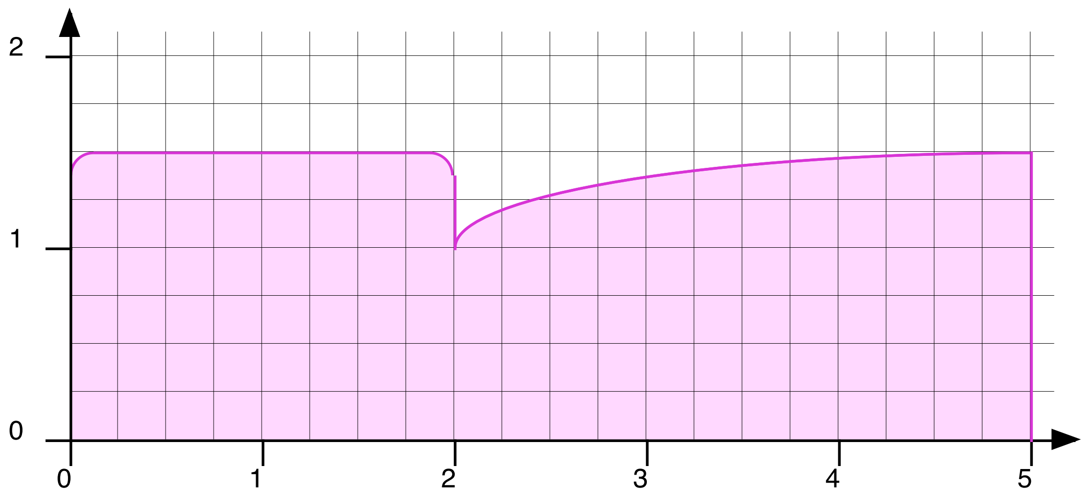
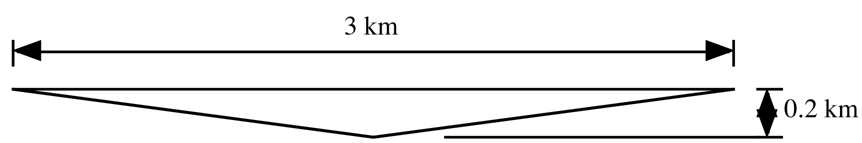


Figure 7.

(e) Evaluate the integral from Part (d) of this problem.

(f) What is the total volume of the champagne cork in units of cubic centimeters? How does this compare to the volume of a conventional cork?

2. A rectangular lake is 150 km long and 3 km wide. The vertical cross-section through the lake is shown in the diagram given below.



(a) Set up an integral for the volume of the lake slicing *vertically*.

(b) Set up an integral for the volume of the lake slicing *horizontally*.

(c) Evaluate the two integrals to find the volume of the lake in cubic kilometers. (You should get the same value each time.)

3. In this problem your objective in each part is to find the volume of revolution when the region described below is rotated around the x-axis. The axis of rotation in each case is the x-axis.

(a) The region bounded by: $y = x^2$, $y = 0$, $x = 0$, and $x = 1$.

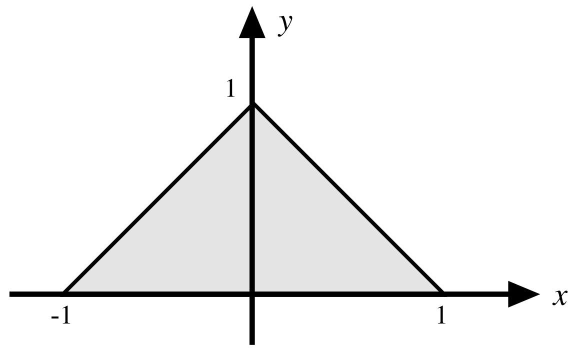
(b) The region bounded by: $y = 4 - x^2$, $y = 0$, $x = -2$, and $x = 0$.

(c) The region bounded by: $y = e^x$, $y = 0$, $x = -1$, and $x = 1$.

(d) The region bounded by: $y = \frac{1}{x+1}$, $y = 0$, $x = 0$, and $x = 1$.

(e) The region bounded by: $y = x^2$, $y = x$, $x = 0$, and $x = 1$.

4. A triangular object is shown below. The density of the object is $\delta(x) = 1 + x$ g/cm^2.

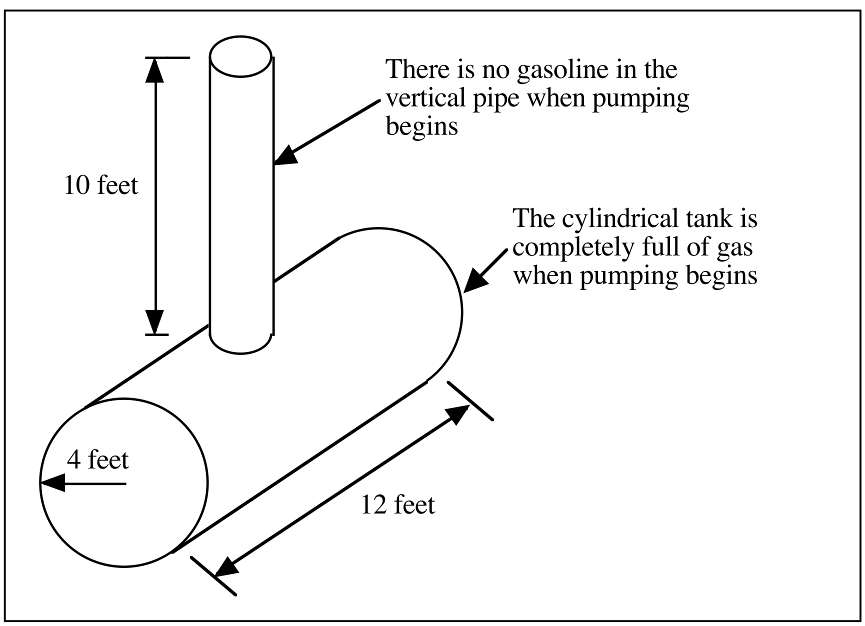


(a) Find the total mass of the object.

(b) Do you think that the *x*-coordinate of the center of mass will be to the left or to the right of the *x*-axis? Briefly (in a sentence or two) explain why you think your answer is correct.

(c) Find the exact value of the *x*-coordinate of the center of mass for the triangular object.

5. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. The radius of the cylinder is 4 feet, its length is 12 feet and its top is 10 feet under the ground.



Let $y = 0$ represent the height at the center of the cylindrical portion of the gasoline tank.

The gasoline stored inside the tank has flakes of rust in it. As a result, the weight of the gasoline varies throughout the tank. The weight of gasoline (in pounds per cubic foot or lb/ft^3) is given by the formula:

$$w(y) = 42 - 0.1y$$

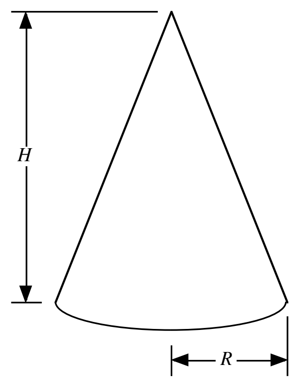
when *y* is measured in feet.

If the tank starts out completely full of gasoline, find the total work needed to pump all of the gasoline out of the tank and up to ground level. In this problem (and this problem only) it is permissible for you to use your calculator to evaluate any definite integrals that you need.

6. An apartment complex stores its salt in a shed that has a conical shape (see diagram below). The height of the shed is H meters and the radius of the base of the shed is R meters. At a vertical height of x meters from the ground, the density of the salt in units of kilograms per cubic meter is given by the function:

$$\delta(x) = k \cdot (H - x),$$

where k is a positive constant.



(a) Assuming that the shed is completely full of salt, find the total mass of the salt in the shed in units of kilograms.

(b) Assuming that the shed is completely full of salt, find the vertical coordinate of the center of mass of the shed.

(c) Suppose that the shed is filled by pouring salt into the shed through a small hole at the top. Salt is moved to the small hole by a horizontal conveyor belt. The shed starts out completely empty and is completely filled with salt. Set up an integral that gives the total amount of work done by gravity as the salt falls off the end of the conveyor belt and settles in the shed.

7. An athlete is participating in an experiment to test a new sports drink. The athlete consumes the sports drink at a steady rate of 300 ml per hour. The athlete continues this rate of consumption for the duration of the experiment. During the experiment, the athlete's body excretes the sports drink at a rate proportional to the amount of sports drink in the athlete's body. Based on experimental measurements, the constant of proportionality is 0.4.

(a) Write down a differential equation for $Q(t)$, the amount (in ml) of sports drink in the athlete's body t hours after the start of the experiment.

(b) Find and classify all equilibrium solutions of the differential equation that you wrote down in Part (a).

(c) Use the technique of Separation of Variables to find a formula for $Q(t)$. You may assume that $Q(0) = Q_0$, a positive constant, and use Q_0 in your final answer.

(d) Suppose that $Q_0 > 0$. What happens to the amount of sports drink in the athlete's body in the long term if the experiment goes on for a long time?

(e) If the athlete starts out with Q_0 ml of sports drink in their body, is it possible to double the amount of sports drink in the athlete's body during the experiment? If so, state the conditions under which this could happen and calculate how long it would take to double the amount of drink in the athlete's body. If not, show why this is not possible.

8. The function $y = f(x)$ is defined by the following differential equation and initial condition:

Differential equation: $f'(x) = x^2 + [f(x)]^2$ or $y' = x^2 + y^2$

Initial value: $f(0) = 1$ or $y(0) = 1$.

(a) Use Euler's Method and $\Delta x = 0.25$ to estimate $f(1)$.

(b) Is the estimate of $f(1)$ that you calculated in Question (a) an over-estimate or an under-estimate of the actual value of $f(1)$? Be careful to explain how you know.

(c) An equilibrium solution of a differential equation is a curve in the *x-y* plane along which the derivative is equal to zero. Does the differential equation in this problem:

Differential equation: $f'(x) = x^2 + [f(x)]^2$ or $y' = x^2 + y^2$

have any equilibrium solutions? If so, find the equation(s). If not, briefly explain why not.

9. In the last twenty years, seafood production has undergone a dramatic change[2]. Natural stocks of fish and crustaceans have dropped dramatically while fish farming (aquaculture) has grown rapidly. Currently, aquaculture accounts for more than 25% of all seafood consumed throughout the world. The two most lucrative species for aquaculture are salmon and shrimp.

In this problem, t will always represent the number of years since 1982, and $P(t)$ will always represent the quantity of shrimp farmed in the world during year t. (The units of $P(t)$ are hundreds of thousands of metric tons.)

The world shrimp production can be represented by the differential equation:

$$\frac{dP}{dt} = -0.1 \cdot P(t) \cdot (P(t) - 7) = -0.1 \cdot P \cdot (P - 7).$$

In 1982, world shrimp production was 100,000 metric tons of shrimp. In symbols: $P(0) = 1$.

(a) Locate the equilibrium solutions of the differential equation given above.

(b) Sketch the slope field for the differential equation given above. Use your slope field to classify the equilibrium solutions found in Part (a).

(c) Use your slope field and axes to sketch a graph showing world shrimp production as a function of time. Perhaps using your graph as a guide, briefly explain the practical significance of each of the equilibrium solutions found in Part (a) in terms of global shrimp production.

(d) Use Euler's method and $\Delta t = 0.5$ to estimate the world shrimp production in the year 1984 ($t = 2$).

(e) Is the estimate of world shrimp production in Part (d) an over- or an under-estimate of the actual level of world shrimp production in 1984? Briefly explain how you know.

---

[2] The data presented in this question was obtained from the paper: R.L Naylor, R.J. Goldburg, H. Mooney, M. Beveridge, J. Clay, C. Folke, N. Kautsky, J. Lubchenco, J. Primavera and M. Williams. (1998) "Nature's subsidies to shrimp and salmon farming." *Science*, **282**: 883-884.

(f) Use the technique of Separation of Variables to find a formula for $P(t)$.

(g) Use the formula from Part (f) to calculate global shrimp production in 1984. Does your answer to this problem support or contradict your answer to Part (e)?

10. **(a)** Find the general solution for the differential equation: $y'' - 3y' - 4y = 0$.

(b) Show that the differential equation $y'' - 3y' - 4y = -8e^t \cos(2t)$ has a solution of the form:

$$y(t) = Ae^t \cos(2t) + Be^t \sin(2t),$$

where A and B are constants. Determine the numerical values of A and B.

(c) Find a solution to the initial value problem:

$$y'' - 3y' - 4y = -8e^t \cos(2t) \qquad y(0) = 1 \qquad y'(0) = -1,$$