Outline

1. Review of trapezoid and midpoint rules.
2. Errors.
3. Simpson's Rule
4. Error in Simpson's rule.

1. Review of Trapezoid and Midpoint

- Trapezoid $= \dfrac{\text{Left R.Sum} + \text{Right R.sum}}{2}$

- Midpoint = Left R. Sum with $\frac{\Delta x}{2}$ inside Y1.

Example

$$p(x) = \frac{1}{15 \cdot \sqrt{2\pi}} e^{-(x-100)^2/450.}$$

Estimate $\int_{115}^{145} p(x)\,dx$ using

(a) Midpoint
(b) Trapezoid

and 100 rectangles.

Solution

(a) Midpoint = 0.157301

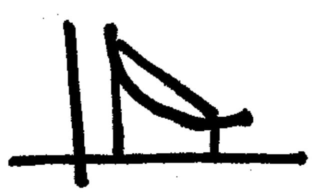
(a) LH Sum = 0.1596883673

RH Sum = 0.1549375898

Trapezoid = 0.1573129785

1. Review of Trapezoid and Midpoint Rules

- Over and under estimates depend on concavity of $f(x)$.

Concavity of $f(x)$	Trapezoid	Midpoint
Concave down on (a,b)	Underestimate of $\int_a^b f(x)\,dx$ 	Overestimate of $\int_a^b f(x)\,dx$
Concave up on (a,b)	Overestimate of $\int_a^b f(x)\,dx$ 	Underestimate of $\int_a^b f(x)\,dx$

Example

Estimate $\int_2^5 \cos(x^2)\,dx$ using 100 rectangles and:
(a) Midpoint (b) Trapezoid rules.

Solution

(a) Midpoint $= 0.1500690351$

(b) Trapezoid $= 0.1498777068$

2. Error Estimates for Midpoint & Trapezoid Rules

- Error = difference between your estimate of $\int_a^b f(x)\,dx$ and the actual value of $\int_a^b f(x)\,dx$.

For trapezoid and midpoint rule, if you estimate $\int_a^b f(x)\,dx$ with N rectangles:

Trapezoid rule:

$$|\text{Error}| \leq \frac{K\cdot(b-a)^3}{12\cdot N^2}$$

Midpoint rule:

$$|\text{Error}| \leq \frac{K\cdot(b-a)^3}{24\cdot N^2}$$

where: K = maximum value of $|f''(x)|$ between $x=a$ and $x=b$.

Example

How many trapezoids do you have to use to approximate $\int_2^5 \cos(x^2)\,dx$

with an error less than 0.0001 ?

Solution

$a = 2 \qquad b = 5 \qquad f(x) = \cos(x^2)$

$\text{Error} < 0.0001 \qquad N = ?$

$$\text{Error} \leq \frac{K \cdot (b-a)^3}{12 \cdot N^2} < 0.0001$$

To find K:

$$f(x) = \cos(x^2)$$

$$f'(x) = -\sin(x^2)\cdot 2x$$

$$f''(x) = -\cos(x^2)\cdot 4x^2 - \sin(x^2)\cdot 2$$

Graph $|f''(x)|$ on calculator to get K.

Use $K = 98.86$.

$$\frac{(98.86)(5-2)^3}{12 \cdot N^2} < 0.0001$$

$$N > \sqrt{\frac{(98.86)(5-2)^3}{(12)(0.0001)}}$$

$$N > 1491.43$$

Use 1492 trapezoids.

3. Simpson's Rule

- Even more accurate method for approximating $\int_a^b f(x)\,dx$ based on areas under parabolas.

- Simplest way to calculate Simpson's Rule approximations:

$$S_{2N} = \frac{1}{3} T_N + \frac{2}{3} M_N$$

Simpson's approximation of $\int_a^b f(x)\,dx$ using 2N rectangles.

Trapezoid rule using N rectangles

Midpoint rule using N rectangles.

Example

Estimate $\int_2^5 \cos(x^2)\,dx$ using Simpson's rule and 200 rectangles.

Solution

$$M_{100} = 0.1500690351$$

$$T_{100} = 0.1498777068$$

$$S_{200} = \frac{1}{3} T_{100} + \frac{2}{3} M_{100}$$

$$= 0.150005259.$$