Outline

1. Polynomial long division.
2. Completing the square.

Review of Partial Fractions

Example

Evaluate: $\int \frac{2x^2-8}{x^3-4x^2+5x-2}\,dx$

Solution

Need to factor:

y1 = x^3 - 4x^2 + 5x - 2

Graph has x-intercept at $x=2$, so $(x-2)$ is a factor.

Dividing the cubic by $x-2$:

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
x-2 & x^3 - 4x^2 + 5x - 2 \\
 & \underline{x^3 - 2x^2} \\
 & \quad -2x^2 + 5x \\
 & \quad \underline{-2x^2 + 4x} \\
 & \qquad\qquad x - 2 \\
 & \qquad\qquad \underline{x - 2} \\
 & \qquad\qquad\quad 0
\end{array}
$$

So:

$$x^3 - 4x^2 + 5x - 2 = (x-2)(x-1)^2$$

Partial fractions:

$$\frac{2x^2-8}{(x-2)(x-1)^2} = \frac{2(x+2)(x-2)}{(x-2)(x-1)^2}$$

$$= \frac{2(x+2)}{(x-1)^2}$$

$$\frac{2x+4}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x+4 = A(x-1) + B$$

$$A = 2 \qquad B = 6$$

<u>So:</u>

$$\int \frac{2x^2-8}{x^3-4x^2+5x-2}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{6}{(x-1)^2}\,dx$$

$$= 2\cdot \ln(|x-1|) + -6\cdot(x-1)^{-1} + C$$

1. Polynomial Long Division

- Good for integrating rational functions where

degree of top $\geqslant$ degree of bottom.

Example

Evaluate: $$\int \frac{x^4 - 8x^3 + 3x^2 + 9x + 1}{x^2 - 4x + 3}\,dx$$

Solution

$$
\begin{array}{r}
x^2 - 4x - 16 \\
x^2 - 4x + 3 \overline{\big)\, x^4 - 8x^3 + 3x^2 + 9x + 1} \\
\underline{x^4 - 4x^3 + 3x^2 } \\
-4x^3 + 0x^2 + 9x \\
\underline{-4x^3 + 16x^2 - 12x } \\
-16x^2 + 21x + 1 \\
\underline{-16x^2 + 64x - 48} \\
-43x + 49
\end{array}
$$

This means:

$$
\frac{x^4 - 8x^3 + 3x^2 + 9x + 1}{x^2 - 4x + 3} = x^2 - 4x - 16 + \frac{-43x + 49}{x^2 - 4x + 3}
$$

So:

$$\int \frac{x^4 - 8x^3 + 3x^2 + 9x + 1}{x^2 - 4x + 3}\,dx$$

$$= \int (x^2 - 4x - 16)\,dx$$

$$+ \int \frac{-43x + 49}{x^2 - 4x + 3}\,dx$$

Use partial fractions for

$$\int \frac{-43x + 49}{x^2 - 4x + 3}\,dx :$$

$$\frac{-43x + 49}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$-43x + 49 = A(x-3) + B(x-1)$$

$$-43x + 49 = Ax - 3A + Bx - B$$

$\underline{x^1:}$ $-43 = A + B$

$\underline{x^0:}$ $49 = -3A - B$

$A = -3 \quad B = -40$

Putting everything together:

$$\int \frac{x^4 - 8x^3 + 3x^2 + 9x + 1}{x^2 - 4x + 3}\, dx$$

$$= \frac{1}{3}x^3 - \frac{4}{2}x^2 - 16x$$

$$-3 \cdot \ln(|x-1|) - 40 \cdot \ln(|x-3|) + C$$

2. Completing the Square

- Integrate rational functions that resemble:

$$\frac{\text{linear function}}{\text{quadratic does not factor}}$$

- Key formula:

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

- Key technique: u-substitution.

How to Complete the Square

Idea: $ax^2 + bx + c$ "Standard Form"

manipulate $\downarrow$

$a(x-h)^2 + k$ "Vertex form"

Example

① $x^2 \boxed{-14} x + 50$

Coeff of $x = -14$
Half $= -7$
Square $= (-7)^2$
$= 49$

② $\underbrace{x^2 - 14x + 49} - 49 + 50$

③ $(x-7)^2 - 49 + 50$

④ $(x-7)^2 + 1$

Example

Evaluate: $\int \frac{3x-4}{x^2-14x+50}\,dx$

Solution

Complete square in denominator

$$\int \frac{3x-4}{x^2-14x+50}\,dx = \int \frac{3x-4}{1+(x-7)^2}\,dx$$

let $u = x-7$

$x = u+7$

$3x-4 = 3u+17$

$$= \int \frac{3u+17}{1+u^2}\,du$$

$$= \int \frac{3u}{1+u^2}\,du + \int \frac{17}{1+u^2}\,du$$

$$= \frac{3}{2}\ln(|1+u^2|) + 17\cdot\arctan(u) + C$$

$$= \frac{3}{2} \ln\left(\left| 1 + (x-7)^2 \right| \right)$$

$$+ 17 \arctan(x-7) + C$$