Outline

1. Idea of partial fractions.

2. 3 cases of partial fractions & how to handle them.

1. Idea of Partial Fractions

e.g. We want to evaluate:

$$\underbrace{\int \frac{\overset{\text{Degree}=1}{5x+11}}{\underset{\text{Degree}=2}{x^2+2x-3}}\,dx}_{\text{Rational function}}$$

- Want antiderivative of a rational function with degree of top < degree of bottom.

To actually do this, note:

$$\frac{1}{x+3} + \frac{4}{x-1} = \frac{4(x+3)+1(x-1)}{(x+3)(x-1)}$$

$$= \frac{5x+11}{x^2+2x-3}$$

So:

$$\int \frac{5x+11}{x^2+2x-3}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{4}{x-1}\,dx$$

$$= \ln(|x+3|) + 4\cdot\ln(|x-1|) + C$$

<u>Question</u>: How do you start with

$$\frac{5x+11}{x^2+2x-3}$$

and break it down into

$$\frac{1}{x+3} + \frac{4}{x-1} ?$$

2. 3 cases of Partial Fractions

How you break the rational function down depends on the denominator of the rational function.

Case 1: The denominator factors completely into unique linear factors.

e.g. $$\frac{5x+11}{x^2+2x-3} = \frac{5x+11}{(x+3)(x-1)}$$

linear factors

Break this down:

$$\frac{5x+11}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Goal: Figure out the numbers A and B.

To do this, add

$$\frac{A}{x+3} \text{ and } \frac{B}{x-1}$$

$$\frac{5x+11}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$\frac{5x+11}{(x+3)(x-1)} = \frac{A\cdot(x-1) + B\cdot(x+3)}{(x+3)(x-1)}$$

Focus on numerators:

$$\boxed{5x} + 11 = \boxed{Ax} - A + \boxed{Bx} + 3B$$

Now equate coefficients of powers of x.

$$5x = Ax + Bx$$

$$11 = -A + 3B$$

Eliminate the x to give:

$$5 = A + B$$

$$11 = -A + 3B$$

Now solve these to get A and B.

Add: $16 = 4B$

$$4 = B$$

Plug into 1st equation: $5 = A + 4$

$$1 = A$$

So:

$$\frac{5x + 11}{(x+3)(x-1)} = \frac{1}{x+3} + \frac{4}{x-1}.$$

Case 2: Denominator factors into linear factors by one or more factors are repeated.

Example

Evaluate: $\int \frac{1}{x^3+2x^2+x}\,dx$

Solution

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)}$$

$$= \frac{1}{x\,(x+1)^2}$$

$(x+1)$ factor
is repeated.

Partial fractions:

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

extra factor needed because
adding these fractions
gives denominator of
$x(x+1)^2$

<u>Goal:</u> Figure out numerical
values of A, B, C.

Add $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$$\frac{1}{x(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$

Focus on numerators:

$$1 = A(x^2 + 2x + 1)$$
$$+ B(x^2 + x)$$
$$+ Cx$$

Equate coefficients:

$\underline{x^2:}$ $\quad 0x^2 = Ax^2 + Bx^2$

$$\boxed{0 = A + B}$$

$\underline{x^1:}$ $\quad 0x = 2Ax + Bx + Cx$

$$\boxed{0 = 2A + B + C}$$

$\underline{x^0:}$ $\quad \boxed{1 = A}$

$$A = 1$$
$$B = -1$$
$$C = -1$$

<u>So:</u>

$$\frac{1}{x(x+1)^2} = \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}$$

$$\int \frac{1}{x(x+1)^2}\,dx = \ln(|x|) - \ln(|x+1|) + (x+1)^{-1} + C$$

Case 3: Denominator does not factor completely.

Example

Evaluate $\int \frac{1}{x^3+x}\,dx$

Solution

$$\frac{1}{x^3+x} = \frac{1}{x\underbrace{(x^2+1)}_{\text{does not factor any further.}}}$$

This time, partial fractions look like:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

<u>Goal:</u> Find A, B, C.

$$1 = A(x^2+1) + x \cdot (Bx+C)$$

Equating coefficients:

<u>x^2:</u> $0x^2 = Ax^2 + Bx^2$

$0 = A + B$

<u>x^1:</u> $0x = Cx$

<u>x^0:</u> $1 = A$

$A = 1$

$B = -1$

$C = 0$

So:

$$\int \frac{1}{x(x^2+1)}dx = \int \frac{1}{x}dx$$

$$- \int \frac{x}{1+x^2}dx$$

$$= \ln(|x|)$$

$$- \frac{1}{2}\ln(|1+x^2|) + C$$