Outline

1. SOHCAH TOA
2. Trigonometric Substitutions.
3. Examples.

1. Example

Evaluate: $\int \sqrt{1-x^2}\, dx$

Solution

① $x = \sin(\theta)$

$$\frac{dx}{d\theta} = \cos(\theta)$$

$$dx = \cos(\theta)\, d\theta$$

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{\underbrace{1-\sin^2(\theta)}_{\cos^2(\theta)}} \cdot \cos\theta \, d\theta$$

$$= \int \cos(\theta) \cdot \cos(\theta)\, d\theta$$

② $$= \int \cos^2(\theta)\, d\theta$$

$$= \int \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

3 $$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

2θ

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$x = \sin(\theta) \qquad \theta = \sin^{-1}(x)$$

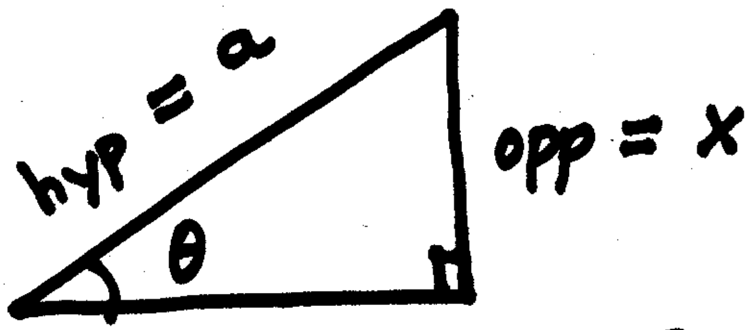
$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}(x)$$

4 $$+ \frac{1}{4}\sin\left(2\cdot\sin^{-1}(x)\right) + C$$

2. SOHCAH TOA

$$X = a \cdot \sin(\theta)$$

$$\sin(\theta) = \frac{X}{a} = \frac{\text{opp.}}{\text{hyp.}}$$



$$\text{adj} = \sqrt{a^2 - X^2} \quad \text{(Pythagoras)}$$

$$\sin(\theta) = \frac{X}{a} \qquad \csc(\theta) = \frac{a}{X}$$

$$\cos(\theta) = \frac{\sqrt{a^2-X^2}}{a} \qquad \sec(\theta) = \frac{a}{\sqrt{a^2-X^2}}$$

$$\tan(\theta) = \frac{X}{\sqrt{a^2-X^2}} \qquad \cot(\theta) = \frac{\sqrt{a^2-X^2}}{X}$$

3. Substitions

(i) When a factor:

$$\sqrt{a^2 - x^2}$$

is in the integral, use:

$$x = a \cdot \sin(\theta)$$

$$dx = a \cdot \cos(\theta)\, d\theta$$

and use identity:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

$$a^2 \cdot \cos^2(\theta) = a^2 - a^2 \cdot \sin^2(\theta)$$

(ii) When a factor:

$$\sqrt{a^2 + x^2}$$

is in the integral, use:

$$x = a \cdot \tan(\theta)$$

$$dx = a \cdot \sec^2(\theta)\, d\theta$$

and use the identity:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$$

$$a^2 + \underbrace{}_{x^2} = a^2\sec^2(\theta)$$

(iii') If a factor of:

$$\sqrt{x^2 - a^2}$$

is present in integral, we:

$$x = a \cdot \sec(\theta)$$

$$dx = a \cdot \sec(\theta)\tan(\theta)\, d\theta$$

and we the identity:

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

$$a^2 \cdot \sec^2(\theta) - a^2 = a^2 \tan^2(\theta)$$

4. Examples

Evaluate: $\displaystyle\int \frac{x^3}{\sqrt{16-x^2}}\,dx$

$\sqrt{a^2-x^2}$ $\quad a=4$

Solution:

① $x = 4\cdot\sin(\theta)$

$dx = 4\cdot\cos(\theta)\,d\theta$

② $$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{64\sin^3(\theta)}{\underbrace{\sqrt{\underbrace{16-16\sin^2(\theta)}_{16\cos^2(\theta)}}}_{4\cdot\cos(\theta)}}\,4\cos(\theta)\,d\theta$$

$$\int \frac{x^3}{\sqrt{16 - \text{[redacted]}\, x^2}}\, dx = \int 64 \sin^3(\theta)\, d\theta$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$= \int 64 \sin(\theta) \cdot \left(1 - \cos^2(\theta)\right) d\theta$$

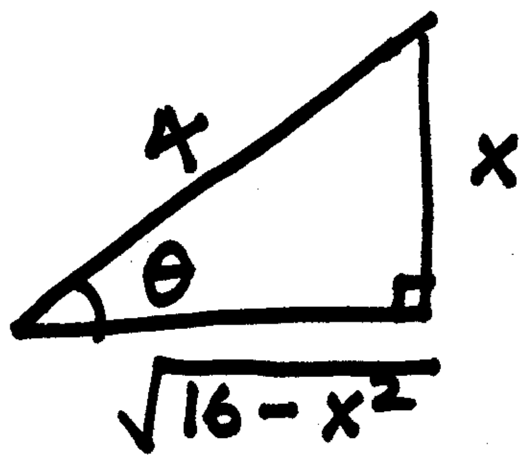
$$= \int 64 \sin(\theta)\, d\theta - \int 64 \sin(\theta) \cos^2(\theta)\, d\theta$$

$u = \cos(\theta)$

$$= -64 \cos(\theta) + \frac{64}{3} \cos^3(\theta) + C$$

$$x = 4 \cdot \sin(\theta)$$

$$\sin(\theta) = \frac{x}{4}$$



$$\cos(\theta) = \frac{adj}{hyp}$$

$$\cos(\theta) = \frac{\sqrt{16 - x^2}}{4}$$

Express final answer in terms of x:

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = -64 \cdot \frac{\sqrt{16-x^2}}{4} + \frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C$$

(iii) <u>Evaluate:</u> $\displaystyle\int \frac{\sqrt{9+x^2}}{x}\,dx$

<u>Solution</u> $\boxed{a=3}$

$$x = 3\cdot\tan(\theta)$$

$$dx = 3\cdot\sec^2(\theta)\,d\theta$$

Substitute into integral:

$$\int \frac{\sqrt{9+x^2}}{x}\,dx = \int \frac{\sqrt{9+9\cdot\tan^2(\theta)}}{3\cdot\tan(\theta)}\,3\cdot\sec^2(\theta)\,d\theta$$

$$= \int \frac{3\cdot\sec(\theta)}{3\cdot\tan(\theta)}\cdot 3\sec^2(\theta)\,d\theta$$

$$= \int \frac{3\cdot\sec^3(\theta)}{\tan(\theta)}\,d\theta$$

$$= \int \frac{3 \cdot \sec^3(\theta)}{\tan(\theta)} \, d\theta$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\frac{3 \cdot \sec^3(\theta)}{\tan(\theta)} = \frac{3}{\cos^3(\theta)} \cdot \frac{\cos(\theta)}{\sin(\theta)}$$

$$= \frac{3 \cdot 1}{\cos^2(\theta) \cdot \sin(\theta)}$$

$$= \frac{3 \cdot \left(\sin^2(\theta) + \cos^2(\theta) \right)}{\cos^2(\theta) \cdot \sin(\theta)}$$

$$= \frac{3 \sin^2(\theta)}{\cos^2(\theta) \sin(\theta)} + \frac{3 \cos^2(\theta)}{\cos^2(\theta) \sin(\theta)}$$

$$= \frac{3 \cdot \sin(\theta)}{\cos^2(\theta)} + \frac{3}{\sin(\theta)}$$

Integrating:

$$\int \frac{3 \cdot \sec^3(\theta)}{\tan(\theta)} \, d\theta$$

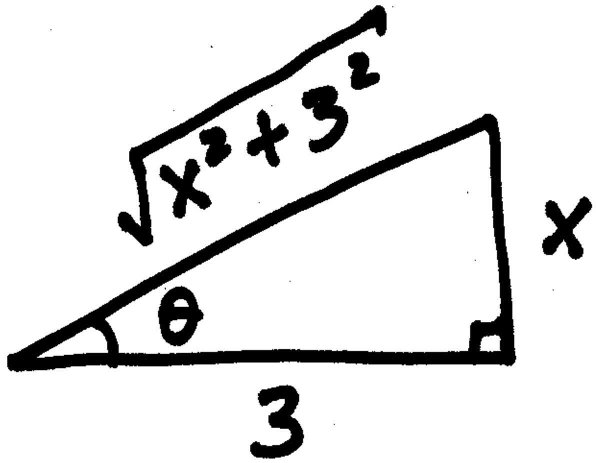
$$= \int \frac{3 \cdot \sin(\theta)}{\cos^2(\theta)} d\theta + \int \frac{3}{\sin(\theta)} d\theta$$

$$= 3 \cdot \frac{1}{\cos(\theta)} + 3 \cdot \ln\left|\csc(\theta) - \cot(\theta)\right| + C$$

Put antiderivative back in terms of x:

$$x = 3 \cdot \tan(\theta)$$

$$\tan(\theta) = \frac{opp}{adj} = \frac{x}{3}$$



$$\cos(\theta) = \frac{3}{\sqrt{x^2+3^2}}$$

$$\csc(\theta) = \frac{\sqrt{x^2+3^2}}{x}$$

$$\cot(\theta) = 3/x$$

<u>So:</u>

$$\int \frac{\sqrt{9+x^2}}{x}\,dx = \frac{3}{\frac{3}{\sqrt{x^2+3^2}}}$$

$$+3 \ln \left| \frac{\sqrt{x^2+3^2}}{x} - \frac{3}{x} \right| + C$$