Outline

1. Parametric equations.
2. Tangent lines.
3. Speed and velocity.
4. Arc length.

Do-over: Thursday 7, 8, 9 pm
2315 DH.

1. Parametric Equations

- Describe a curve in the xy-plane using a pair of functions:

  $x(t)$ = x-coordinate of point on the curve at time t.

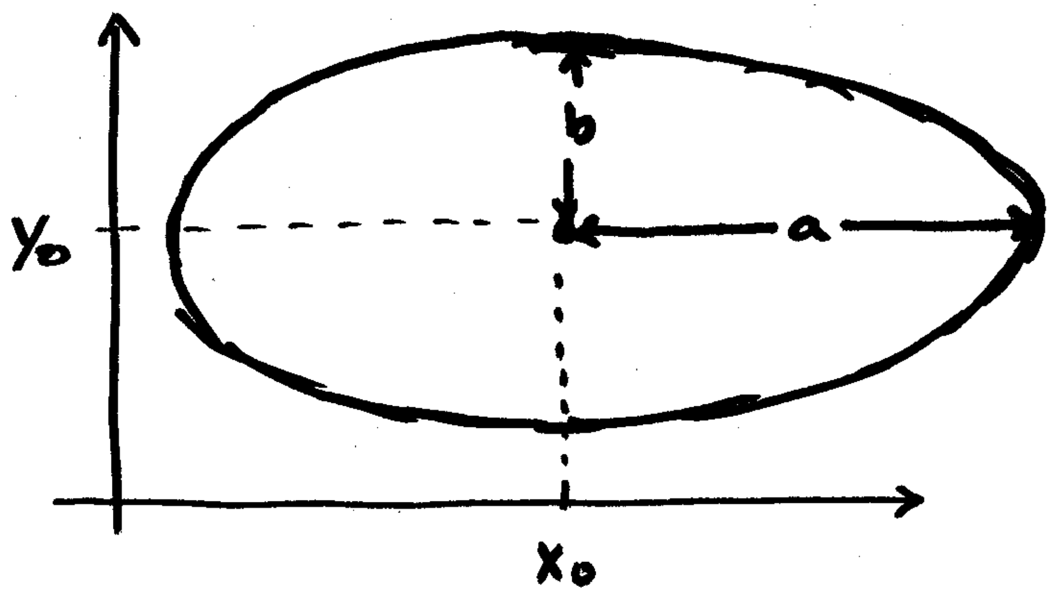
  $y(t)$ = y-coordinate of point on the curve at time t.

Example

- Usual equation for ellipse:

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1.$$

  - Center: (x_0, y_0)
  - Semi-major axis: $\max(a,b)$
  - Semi-minor axis: $\min(a,b)$



$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

- Parametric Equations for Ellipse:

$$x(t) = x_0 + a \cdot \cos(t)$$
$$y(t) = y_0 + b \cdot \sin(t)$$

Interval : $[0, 2\pi]$

2. Tangent Lines

- If we have a curve in the xy-plane described by $y = f(x)$, the equation of the tangent line at the point $(x_0, f(x_0))$ is:

$$y - f(x_0) = f'(x_0) \cdot \left[x - x_0 \right]$$

slope of tangent line $= \dfrac{dy}{dx}$

- To find dy/dx for a curve describe using parametric equations $x(t)$ and $y(t)$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Example

Consider the curve described by:

$$x(t) = \overset{y(t)^2}{t} - \cos(\overset{y(t)^2}{t})$$

$$y(t) = \sqrt{t} \qquad y(t)^2 = t$$

$0 \leq t \leq 4\pi$. These are the parametric equations for the curve defined by:

$$x = y^2 - \cos(y^2).$$

Find the equation for the tangent line to this curve at the point $\left(\pi/2, \sqrt{\pi/2}\right)$.

Solution (Parametric Equations)

- Figure out value the value of t that corresponds to $(\pi/2, \sqrt{\pi/2})$

$$x(t) = t - \cos(t) = \pi/2$$
$$y(t) = \sqrt{t} = \sqrt{\pi/2}$$

So: $\boxed{t = \pi/2}$

- Calculate formula for dy/dx.

$$y'(t) = \frac{1}{2} t^{-1/2}$$
$$x'(t) = 1 + \sin(t)$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{1}{2} t^{-1/2}}{1 + \sin(t)}$$

- Plug $t = \pi/2$ into dy/dx to get slope of tangent line.

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{\frac{1}{2}\left(\pi/2\right)^{-1/2}}{1+\sin\left(\pi/2\right)}$$

$$= \frac{1}{2\sqrt{2\pi}}$$

- Use point-slope form to write equation of tangent line.

$$y - \sqrt{\pi/2} = \frac{1}{2\sqrt{2\pi}} \cdot \left(x - \pi/2\right).$$

<u>Solution (Implicit Differentiation)</u>

$$x = y^2 - \cos(y^2)$$

- Take derivative with respect to x:

$$1 = 2y \cdot \frac{dy}{dx} + \sin(y^2) \cdot 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y + 2y \cdot \sin(y^2)}$$

- To find slope plug in
  $x = \pi/2, \quad y = \sqrt{\pi/2}$:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\pi/2,\sqrt{\pi/2})} = \frac{1}{2\sqrt{\pi/2} + 2\sqrt{\pi/2}\,\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{4\sqrt{\pi/2}} = \frac{1}{2\sqrt{2\pi}}$$

- Use point-slope form:

$$y - \sqrt{\pi/2} = \frac{1}{2\sqrt{2\pi}}\left(x - \pi/2\right)$$

3. Speed and Velocity

- Imagine an object moving in the xy-plane, its location at time t given by $x(t)$ and $y(t)$, then:

  - Velocity $= \vec{v}(t)$

  $$= \langle x'(t), y'(t) \rangle$$

  - Speed $= s(t)$

  $$= \| \vec{v}(t) \|$$

  $$= \sqrt{(x'(t))^2 + (y'(t))^2}$$

4. Arc Length

- A curve is described by the parametric equations $X(t)$ and

$y(t)$. The length of the curve between $t=a$ and $t=b$ is:

speed

$$\text{Arc length} = \int_a^b s(t)\,dt$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$