Outline

1. Error estimates for approximating functions with Taylor polynomials.

Exam: Friday during lecture.

Do-over: 12/4/08 7pm, 8pm, 9pm
2315 DH.

Final: 12/12/08 8:30am – 11:30am.

1. Error from Approximating a Function with a Taylor Polynomial

- Let $f(x)$ be a function.
- Let $P_N(x)$ be the N^{th} degree Taylor polynomial of $f(x)$ with center at $x = a$.
- If $x = b$ is a given point and we use $P_N(b)$ to approximate $f(b)$ then the error is:

$$\text{Error} \leq \frac{M}{(N+1)!} \cdot |b-a|^{N+1}$$

where M is the maximum value of $|f^{(N+1)}(x)|$ over the interval $[a, b]$.

Example

(a) Calculate an approximate value of $\sin(2)$ using degree 6 Taylor polynomial of $f(x) = \sin(x)$ centered at $a = \pi/2$.

(b) Find the error in this approximation.

Solution

(a) Create a formula for

$P_6(x)$.

$$f(x) = \sin(x) \qquad a = \pi/2$$

$$\begin{aligned}
f(x) &= \sin(x) & f(\pi/2) &= 1 \\
f'(x) &= \cos(x) & f'(\pi/2) &= 0 \\
f''(x) &= -\sin(x) & f''(\pi/2) &= -1 \\
f'''(x) &= -\cos(x) & f'''(\pi/2) &= 0 \\
f^{(iv)}(x) &= \sin(x) & f^{(iv)}(\pi/2) &= 1 \\
f^{(v)}(x) &= \cos(x) & f^{(v)}(\pi/2) &= 0 \\
f^{(vi)}(x) &= -\sin(x) & f^{(vi)}(\pi/2) &= -1.
\end{aligned}$$

$$P_6(x) = 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - \frac{1}{6!}(x-\pi/2)^6$$

To approximate $\sin(2)$ plug $x = b = 2$ into $P_6(x)$.

$$\sin(2) \approx 1 - \frac{1}{2!}\left(2-\pi/2\right)^2 + \frac{1}{24}\left(2-\pi/2\right)^4$$

$$- \frac{1}{720}\left(2-\pi/2\right)^6$$

$$= 0.9092973983$$

(b) $f(x) = \sin(x)$ $\qquad a = \pi/2$

$N = 6$ $\qquad b = 2$

$$\text{Error} \leq \frac{M}{7!} \cdot \left|2 - \pi/2\right|^7$$

$$f^{(vii)}(x) = -\cos(x)$$

$$\left| f^{(vii)}(x) \right| = \left|\cos(x)\right|$$

M = maximum value of $|\cos(x)|$ between $x = a = \pi/2$ and $x = b = 2$.

$$= 0.416146836547$$

$$\text{Error} \le \frac{0.41614683654 7}{7!}\left|2-\pi/2\right|^{7}$$

$$= 2.2154 \times 10^{-7}$$

Our approximation:

$\sin(2) \approx 0.\underline{909297}3983$

Calculator value:

$\sin(2) \approx 0.\underline{909297}4268$