Outline

1. More Taylor series creation methods.
2. Interval of convergence.
3. Accuracy of Taylor polynomial approximations.

1. Creating New Taylor Series

① $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad a = 0.$$

② $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad a = 0.$$

③ $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad a = 0.$$

④ $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad a = 0.$$

⑤ $$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1} \qquad a = 0.$$

⑥ Binomial Theorem:

$$(1+x)^p = 1 + p \cdot x + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots$$

Example

Find the Taylor series of

$f(x) = \dfrac{1}{\sqrt{1-x^2}}$ centered at $a=0$.

Solution

Binomial Theorem:

$$(1+\square)^p = 1 + p\square + \frac{p(p-1)}{2!}\square^2 + \frac{p(p-1)(p-2)}{3!}\square^3 + \cdots$$

$$f(x) = \frac{1}{\sqrt{1-x^2}} = \left(1 + \boxed{-x^2}\right)^{-1/2}$$

$p = -1/2$ $\qquad\qquad$ $\square = -x^2$

Binomial Theorem:

$$f(x) = \left(1 + -x^2\right)^{-1/2}$$

$$= 1 + \left(\frac{-1}{2}\right)\left(-x^2\right)$$

$$+ \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(-x^2\right)^2$$

$$+ \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(-x^2\right)^3 + \cdots$$

<u>Example</u>

Find the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $a = 5$.

<u>Solution</u>

<u>Goal:</u> Power series for $f(x)$ that looks like:

$$f(x) = \sum_{n=0}^{\infty} C_n \cdot (x-5)^n$$

$$\frac{1}{1-\boxed{}} = \sum_{n=0}^{\infty} \boxed{}^{\,n}$$

$$f(x) = \frac{1}{1 - x}$$

Giveth and taketh away 5.

$$= \frac{1}{1-5 - x + 5}$$

$$= \frac{1}{-4 - (x-5)}$$

$$= \frac{-1}{4} \cdot \frac{1}{1 - \boxed{\frac{-1}{4}(x-5)}}$$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} \boxed{\left(\frac{-1}{4}(x-5)\right)}^{n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^{n+1}} \cdot (x-5)^n$$

2. Interval of Convergence

- Power series: $\sum_{n=0}^{\infty} C_n (x-a)^n$
- Radius of convergence: r
- Interval of convergence:
  - The set of all x-values that can be plugged in to the power series to give a convergent infinite series.
- Consists of the interval

$$(a-r, \; a+r)$$

and maybe the endpoints $x = a-r$ and $x = a+r$.

Example

Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n} \cdot (x+7)^n$.

Solution

$a = -7$.

Radius of Convergence, r

$$a_n = \frac{(-1)^n \cdot n}{4^n} \cdot (x+7)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{-1 \cdot (n+1)}{4n} \cdot (x+7)$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4} \cdot |x+7| < 1$$

$$|x+7| < 4$$

Radius of convergence $= r = 4$.

Interval of convergence includes:

$$(-7-4\ ,\ -7+4)$$

$$(-11\ ,\ -3)$$

Check endpoint $x=-11$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n}(-11+7)^n = \underbrace{\sum_{n=0}^{\infty} n}$$

divergent so $x=-11$ not in interval.

Check endpoint $x=-3$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n}(-3+7)^n = \underbrace{\sum_{n=0}^{\infty} (-1)^n \cdot n}$$

divergent so $x=-3$ not in interval.

Interval of convergence:

$$(-11,\ -3).$$