Outline.

1. Finding a Taylor Series.

2. Radius of convergence.

Test #3: Friday 11/21/08

Do-Over: Thursday 12/4/08.

1. Finding a Taylor Series

- Definition of the Taylor series of $f(x)$ centered at $x = a$ is:

Taylor **series** because it is an infinite sum.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

- The Taylor polynomial of degree N is the finite series:

Taylor polynomial because it is a finite sum.

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(a) Direct Calculation

- Calculate $f(a)$, $f'(a)$, $f''(a)$, etc. and plug into the Taylor series / polynomial definition.

Example.

Find the degree 6 Taylor Polynomial of $f(x) = \cos(x)$ with $a = 2\pi$.

Solution

$f(x) = \cos(x)$ $\qquad$ $f(2\pi) = 1.$

$f'(x) = -\sin(x)$ $\qquad$ $f'(2\pi) = 0.$

$f''(x) = -\cos(x)$ $\qquad$ $f''(2\pi) = -1$

$f'''(x) = \sin(x)$ $\qquad$ $f'''(2\pi) = 0$

$f^{(iv)}(x) = \cos(x)$ $\qquad$ $f^{(iv)}(2\pi) = 1$

$$f^{(v)}(x) = -\sin(x) \qquad f^{(v)}(2\pi) = 0$$

$$f^{(vi)}(x) = -\cos(x) \qquad f^{(vi)}(2\pi) = -1.$$

Taylor polynomial $= 1 + \frac{0}{1!}(x-2\pi)^1$

$$+ \frac{-1}{2!}(x-2\pi)^2$$

$$+ \frac{0}{3!}(x-2\pi)^3$$

$$+ \frac{1}{4!}(x-2\pi)^4$$

$$+ \frac{0}{5!}(x-2\pi)^5$$

$$+ \frac{-1}{6!}(x-2\pi)^6$$

Guess for Taylor <u>series</u>:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot (x-2\pi)^{2n}$$

(b) Adapting a Known Taylor Series

- Adaptation methods include:
  - Taking a derivative or integral.
  - Adding or subtracting power series
  - Multiplying a series by a certain factor.
  - Strategically "Giveth and Taketh away" a constant. (Good when you want to change the "a" that the series is based at.)

Good Taylor Series to Know:

① $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n \qquad \boxed{a=0}$$

② $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \boxed{a=0}$$

③ $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \boxed{a=0}$$

④ $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \boxed{a=0}$$

$$\frac{1}{1-\boxed{x}} = \sum_{n=0}^{\infty} \boxed{x}^n$$

Example

Find the Taylor series for $f(x) = \frac{1}{1+x}$ with $a=0$.

Solution

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

Example

Find the Taylor series for $g(x) = \ln(1+x)$ with $a = 0$.

Solution

(Don't worry about $+C$):

$$\int \frac{1}{1+x}\,dx = \ln(1+x)$$

To get Taylor series for $g(x)$ integrate $\sum_{n=0}^{\infty} (-1)^n \cdot x^n$.

$$\text{Taylor series for } g(x) = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^n \cdot dx$$

Example

Find Taylor series for $g(x) = e^{-x}$ with $a = 0$.

Solution

$$e^{-x} = e^{(-x)} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot x^n$$

Example

Find Taylor series for:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

with $a=0$.

Solution

$$\sinh(x) = \frac{1}{2} e^{x} - \frac{1}{2} e^{-x}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} x^{n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{n!} x^{n} - \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{(-1)^{n}}{n!} x^{n}$$

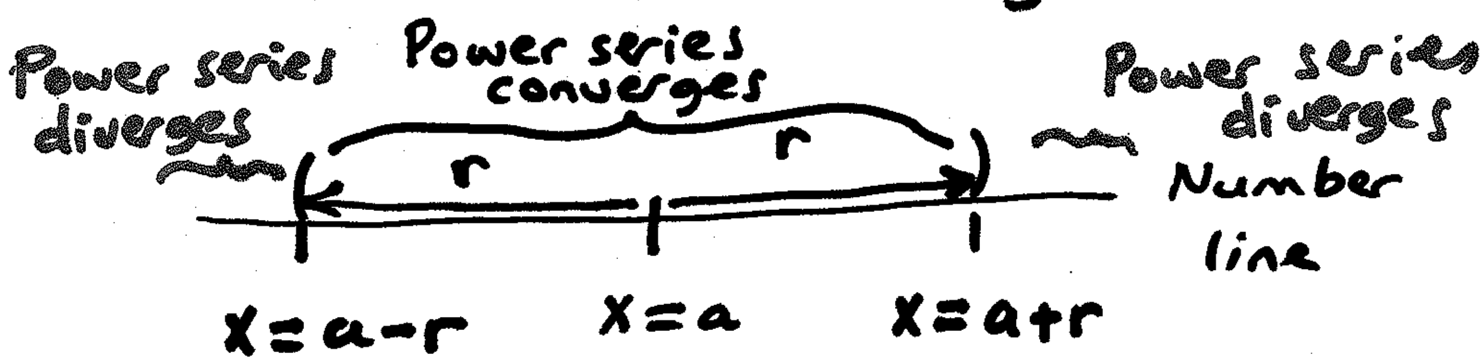
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot \frac{1}{n!} \cdot x^{n} - \frac{1}{2} \cdot \frac{(-1)^{n}}{n!} \cdot x^{n} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{n!} \cdot \left(1 - (-1)^{n} \right) \cdot x^{n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \cdot x^{2n+1}$$

2. Radius of Convergence

- Power series converges at $x = a$ no matter what.
- The distance, r, that you can go away from $x=a$ and still get convergence of the power series is the radius of convergence.



Example

Find the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{7^n} \cdot (x+4)^n .$$

Solution

- Use Ratio Test.

$$a_n = \frac{(-1)^n}{7^n} \cdot (x+4)^n \qquad a_{n+1} = \frac{(-1)^{n+1}}{7^{n+1}} \cdot (x+4)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{7^{n+1}} \cdot (x+4)^{n+1} \cdot \frac{7^n}{(-1)^n} \cdot \frac{1}{(x+4)^n}$$

$$= \frac{-1}{7} \cdot (x+4)$$

- Calculate $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

$$\lim_{n\to\infty} \left| \frac{-1}{7}(x+4) \right| = \frac{1}{7} \cdot |x+4|.$$

- Set limit to be less than 1 and solve for $|x-a|$.

$$\frac{1}{7}|x+4| < 1$$

$$|x+4| < 7$$ Radius of convergence

- So any x-value between

$x = -4 - 7 = -11$ and

$x = -4 + 7 = 3$ will give

a convergent infinite series when plugged into:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{7^n} \cdot (x+4)^n .$$

Example

Find the radius of convergence of :

$$\sum_{n=0}^{\infty} \frac{(n+3)}{5^n} \cdot (x-7)^n$$

Solution

- Use the Ratio Test :

$$a_n = \frac{(n+3)}{5^n} \cdot (x-7)^n \qquad a_{n+1} = \frac{(n+4)}{5^{n+1}} (x-7)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+4)}{5^{n+1}} (x-7)^{n+1} \cdot \frac{5^n}{(n+3)} \cdot \frac{1}{(x-7)^n}$$

$$= \frac{(n+4)}{(n+3)} \cdot \frac{1}{5} \cdot (x-7)$$

Now calculate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+4)}{(n+3)} \cdot \frac{1}{5} \cdot |x-7|$$

$$= \frac{1}{5} \cdot |x-7|$$

To give convergence, force this to be less than 1 and make $|x-a|$ the subject.

$$\frac{1}{5}|x-7| < 1$$

$$|x-7| < 5$$

the radius of convergence.

- At the very least, all x-values from $7-5=2$ up to $7+5=12$ will give convergence if plugged into power series