Outline

1. $\int uv'dx = uv - \int vu'dx$
2. Trig formulas.
3. Trig integrals.

1. Integration by Parts

Example

Evaluate $\int \tan^{-1}(x)\,dx$.

Note: $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$

Solution

$$\int \tan^{-1}(x) \cdot 1\,dx$$

$u = \tan^{-1}(x)$ $\qquad$ $v' = 1$

$u' = \dfrac{1}{1+x^2}$ $\qquad$ $v = x$

$$\int \tan^{-1}(x)\,dx = x \cdot \tan^{-1}(x) - \underbrace{\int \frac{x}{1+x^2}\,dx}_{\substack{u\text{-subst.} \\ u = 1+x^2}}$$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u} \cdot \frac{du}{2x}$$

$$= \frac{1}{2} \int \frac{1}{u}\,du$$

$$= \frac{1}{2} \ln(|u|) + C$$

$$= \frac{1}{2} \ln(1+x^2) + C$$

$$\int \tan^{-1}(x)\,dx = x \cdot \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C$$

2. Trig Formulas

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x).$$

$$\cos(2x) = 2 \cdot \cos^2(x) - 1$$

$$= 1 - 2\sin^2(x),$$

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Example

$\int_0^{\pi/2} \sin^2(x)\,dx$. Evaluate.

Solution

$$\int_0^{\pi/2} \sin^2(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\left(1 - \cos(2x)\right)dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\,dx - \frac{1}{2}\int_0^{\pi/2} \cos(2x)\,dx$$

$u = 2x$

$$= \left[\frac{1}{2}x\right]_0^{\pi/2} - \left[\frac{1}{4}\sin(2x)\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x).$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

3. Trig Integrals

Trick: Use

$$\sin^2(x) + \cos^2(x) = 1$$

to facilitate u-substitution.

Example

$\int \cos^5(x)\, dx$. Evaluate.

Solution

$$\int \cos^5(x)\,dx = \int \cos(x) \cdot \underbrace{\left(\cos^2(x)\right)^2}_{\cos^4(x)} dx$$

$$= \int \cos(x) \cdot \left(1 - \sin^2(x)\right)^2 dx$$

$$= \int \cos(x) \cdot \left(1 - 2\sin^2(x) + \sin^4(x)\right) dx$$

$$= \int \cos(x)\,dx - 2\int \cos(x)\sin^2(x)\,dx$$

$u = \sin(x)$

$$+ \int \cos(x)\sin^4(x)\,dx$$

$u = \sin(x)$

$$= \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$$

Trick: Use

$$\tan^2(x) + 1 = \sec^2(x)$$

to facilitate u-subst.

Example

$\int \tan^3(x) \cdot \sec^9(x)\,dx$. Evaluate.

Solution

$$\int \tan^3(x)\sec^9(x)\,dx = \int \sec(x)\tan(x) \cdot \underbrace{\tan^2(x)}_{\sec^2(x)-1}\sec^8(x)\,dx$$

$$= \int \sec(x)\tan(x)\left(\sec^2(x)-1\right)\sec^8(x)\,dx$$

$$= \int \sec(x)\tan(x)\cdot\sec^{10}(x)\,dx$$

$u = \sec(x)$

$$- \int \sec(x)\tan(x)\cdot\sec^8(x)\,dx$$

$u = \sec(x)$

$$= \frac{1}{11}\sec^{11}(x) - \frac{1}{9}\sec^9(x) + C$$

Trick: Use double angle formulas to eliminate factors of $\sin^2(x)$ or $\cos^2(x)$.

Example

Evaluate: $\int x \cdot \sin^2(x)dx$.

Solution

$$\int x \cdot \sin^2(x)dx = \int x \cdot \tfrac{1}{2}\big(1 - \cos(2x)\big)dx$$

$$= \int \tfrac{1}{2}x\,dx - \int \tfrac{1}{2}x\cos(2x)\,dx$$

$$= \int \frac{1}{2} x\,dx - \int \frac{1}{2} x \cdot \cos(2x)\,dx$$

$$u = x \qquad v' = \cos(2x)$$

$$u' = 1 \qquad v = \frac{1}{2}\sin(2x)$$

$$\int \frac{1}{2} x \cdot \cos(2x)\,dx = \frac{1}{2} \cdot x \cdot \frac{1}{2}\sin(2x)$$

$$- \frac{1}{2} \cdot \int \frac{1}{2} \sin(2x)\,dx$$

$$= \frac{1}{4} x \sin(2x) + \frac{1}{8} \cos(2x)$$

Putting everything together:

x in here.

$$\int x \cdot \sin^2(x)\,dx = \frac{1}{4} x^2 - \frac{1}{4} x \sin(2x)$$

$$- \frac{1}{8} \cos(2x) + C$$