Outline

1. Comparison test.
2. Elements of style.
3. Alternating series.
4. Alternating series test for convergence.

1. Comparison Test

If $0 \leq a_n \leq b_n$ then:

(I) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.

(II) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ also diverges.

Example

Do the following series:

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+2n+3}$ (b) $\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$

converge or diverge?

Intuitive Answers

(a) When n is large

term w/ biggest power of n

$$\frac{n^2+1}{n^4+2n+3} \approx \frac{n^2}{n^4} = \frac{1}{n^2}$$

term with biggest power of n

and $\sum_{n=1}^{\infty} 1/n^2$ (p-series $p=2>1$)

converges, so guess that

$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+2n+3}$ also converges.

(b) If n is large,

$$\frac{6n^2+1}{2n^3-1} \approx \frac{6n^2}{2n^3} = \frac{3}{n}.$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n}$ is a p-series

with $p=1$ so it diverges.

Guess that $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$ also

diverges.

2. Elements of Style

Example (a)

① Up front: Do you think the series converges or diverges?

Guess that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+2n+3}$ converges because for large n,

$$\frac{n^2+1}{n^4+2n+3} \approx \frac{n^2}{n^4} = \frac{1}{n^2},$$

and $\sum_{n=1}^{\infty} 1/n^2$ (a p-series with $p=2$) converges.

② State the convergence test that you will use.

To ~~determine~~ confirm the convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+2n+3}$ we will use the Comparison Test.

③ Show that any conditions the test needs are actually met.

Set $a_n = \frac{n^2+1}{n^4+2n+3}$ then we need to find b_n so that:

(I) $0 \leq a_n \leq b_n$

(II) $\sum_{n=1}^{\infty} b_n$ converges.

④ Carry out the calculations mandated by the convergence test.

Need to create a b_n that satisfies (I) and (II).

Strategy : Start with denominator of a_n.

$$n^4 \leq n^4 + 2n + 3$$

So $$\frac{1}{n^4} \geq \frac{1}{n^4 + 2n + 3}$$

$$\frac{n^2+1}{n^4} \geq \frac{n^2+1}{n^4 + 2n + 3}$$

Since $n^2 \geq 1$ when $n \geq 1$

$$\frac{n^2 + n^2}{n^4} \geq \frac{n^2+1}{n^4} \geq \frac{n^2+1}{n^4+2n+3}$$

So overall:

$$\frac{2n^2}{n^4} \geq \frac{n^2+1}{n^4+2n+3}$$

Set: $b_n = \dfrac{2n^2}{n^4}$.

Now $0 \leq \dfrac{n^2+1}{n^4+2n+3} \leq \dfrac{2n^2}{n^4}$

means that $0 \leq a_n \leq b_n$
so Condition (I) is met.

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges (p-series
with $p=2$) so Condition (II)
is met.

5 Conclusion.

By the comparison test,

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+2n+3} \text{ converges.}$$

2. Elements of Style

Example (b)

Does $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$ converge or diverge?

Solution

① Statement of what your intuition says about convergence or divergence.

For large values of n,

$$\frac{6n^2+1}{2n^3-1} \approx \frac{6n^2}{2n^3} = \frac{3}{n}.$$

The series $\displaystyle\sum_{n=1}^{\infty} 3/n$ is a p-series with $p=1$, so it diverges.

We guess that $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$

also diverges.

② State the Convergence Test used.

To confirm the divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$ we will use the Comparison test.

③ Demonstrate that any preconditions that the Convergence test requires are met.

Let $b_n = \dfrac{6n^2+1}{2n^3-1}$.

We need to find a_n so that:

(I) $\quad 0 \leq a_n \leq b_n$

(II) $\quad \sum_{n=1}^{\infty} a_n$ diverges.

④ Carry out computations needed to do convergence test.

Here we build a_n and use a_n to verify conditions (I) and (II).

<u>Strategy:</u> Start with denominator of b_n

$$2n^3 - 1 \leq 2n^3$$

So:

$$\frac{1}{2n^3 - 1} \geqslant \frac{1}{2n^3}$$

Numerators:

$$6n^2 + 1 \geqslant 6n^2$$

and:

$$\frac{6n^2+1}{2n^3-1} \geqslant \frac{6n^2}{2n^3} = \frac{3}{n}.$$

Let $a_n = 3/n$. We have shown:

$$0 \leq a_n \leq b_n$$

$$0 \leq 3/n \leq \frac{6n^2+1}{2n^3-1}$$

So Condition (I) is met.

For condition (II),

$$\sum_{n=1}^{\infty} \frac{3}{n} = \sum_{n=1}^{\infty} a_n$$

and $\sum_{n=1}^{\infty} 3/n$ is a p-series

with $p=1$, so it diverges.

So Condition (II) is met.

⑤ Conclusion.

By the Comparison test,

$$\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1} \text{ diverges.}$$