Outline

1. Convergence and Divergence.
2. n^{th} term test.
3. Integral test.
4. Ratio test.

1. Convergence and Divergence

- **Convergence**: $\sum_{k=1}^{\infty} a_k$ adds up to a finite total.
- **Divergence**: $\sum_{k=1}^{\infty} a_k$ either fails to add up or adds up to an infinite total.

Example

Does the infinite series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k}$$

converges or diverges?

Solution

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

$A = 1 \qquad B = -1$ so:

$$\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$$

Partial Sums:

$$S_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = \frac{1}{1} \underbrace{- \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3}}_{\text{telescoping sum}} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$S_N = 1 - \frac{1}{N+1}$$ N^{th} partial sum.

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

$\lim_{N\to\infty} S_N$ exist and is finite

so $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ converges.

2. n^{th} Term Test for Divergence

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series.

① Compute $\lim_{n\to\infty} a_n$. formula from Σ notation.

② If $\lim_{n\to\infty} a_n \neq 0$ then the infinite series $\sum_{n=1}^{\infty} a_n$ must diverge.

- This test tells you nothing about the convergence of $\sum_{n=1}^{\infty} a_n$.

3. The Integral Test.

- Can tell you about convergence and divergence.
- To do the integral test with an infinite series $\sum_{n=1}^{\infty} a_n$!

① Take formula from Σ notation and write it in $f(x)$ style.

② $f(x)$ has to be:

- ☐ Positive $f(x) \geq 0$ when $x \geq 1$.
- ☐ Decreasing $f'(x) < 0$ when $x \geq 1$.

③ The convergence/divergence of $\sum_{n=1}^{\infty} a_n$ is exactly the same as the convergence / divergence of $\int_1^{\infty} f(x)\,dx$ (improper integral)

Example

Do the following series converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

(b) $\displaystyle\sum_{n=1}^{100} \frac{1}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$?

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$ $\qquad a_n = \frac{n}{n+2}.$

nth term: $\displaystyle\lim_{n\to\infty} \frac{n}{n+2} = 1$

Since $\lim_{n\to\infty} a_n = 1 \neq 0$,

n^{th} term test gives that

$\sum_{n=1}^{\infty} \frac{n}{n+2}$ diverges.

finite series

(b) $\sum_{n=1}^{100} \frac{1}{n}$. Converges

because it is a finite series.

(c) $\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad a_n = \frac{1}{n}$.

n^{th} term: $\lim_{n\to\infty} \frac{1}{n} = 0$.

n^{th} term test gives no information. Try integral test.

Integral test:

$a_n = \frac{1}{n}$ $\qquad f(x) = \frac{1}{x}$

- <u>Positive:</u> $x > 1$

$$f(x) = \frac{1}{x} \quad \frac{\text{positive}}{\text{poritive}} = +$$

- <u>Decreasing:</u> $x > 1$

$$f'(x) = \frac{-1}{x^2} \quad \frac{\text{negative}}{\text{positive}} = -$$

decreasing because $f'(x) < 0$.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx$$

$$= \lim_{a\to\infty} \Big[\ln(|x|) \Big]_1^a$$

$$= \lim_{a\to\infty} \ln(|a|) - \ln(1)$$

$$= +\infty.$$

Because $\int_1^\infty \frac{1}{x}\,dx$ diverges, the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

4. The Ratio Test

- Can tell you about convergence and divergence.
- To do the ratio test with $\sum_{n=1}^{\infty} a_n$:

① Write down a formula for:

$$\frac{a_{n+1}}{a_n}.$$

② Calculate limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

③ Interpret the result.

$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\|$	Interpretation
limit < 1.	$\sum_{n=1}^{\infty} a_n$ converges. (absolutely)
limit > 1	$\sum_{n=1}^{\infty} a_n$ diverges.
limit $= 1$	No information on convergence or divergence.