SOLUTIONS

Math 122 **Fall 2008**

Handout 12: Review Problems for Unit Test 2

The topics that will be covered on Unit Test 2 are as follows.

- Areas between curves.
- Calculating volumes (non-rotation) using integrals.
- Volumes of revolution – disk method.
- Volumes of revolution – washer method.
- Volumes of revolution – shell method.
- Arc length.
- Using integrals to calculate masses.
- Center of mass.
- Using integrals to calculate work in physics.
- Using integrals to calculate hydrostatic force.
- Euler's method.
- Slope fields.
- Equilibrium solutions.
- Separation of variables.
- Integrating factors.
- Second order, homogeneous differential equations with constant coefficients.
- Method of Undetermined Coefficients.

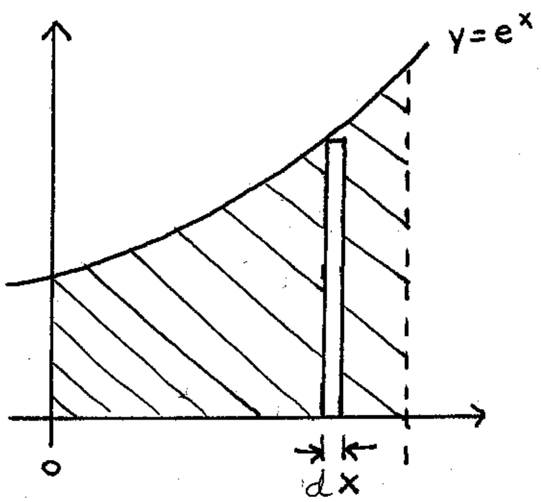
This (roughly) covers Chapter 7 of the textbook plus the additional topics on differential equations that have been covered in lecture and recitation.

1. Calculate the volume of the solid whose base is the region bounded by:

- $y = e^x$,
- the x-axis, and,
- the lines $x = 0$ and $x = 1$

and whose cross-sections are equilateral triangles that are perpendicular to the x-axis.

The base of the region looks like the diagram shown below. The base of one triangle is shown as a rectangle of thickness dx.



The volume of that triangle will be:

$$\text{Triangle volume} = \frac{\sqrt{3}}{4} y^2 \, dx = \frac{\sqrt{3}}{4} e^{2x} \, dx$$

The total volume is:

$$\text{Volume} = \int_0^1 \frac{\sqrt{3}}{4} e^{2x} \, dx = \left[\frac{\sqrt{3}}{8} e^{2x}\right]_0^1 = \frac{\sqrt{3}}{8}(e^2 - 1)$$

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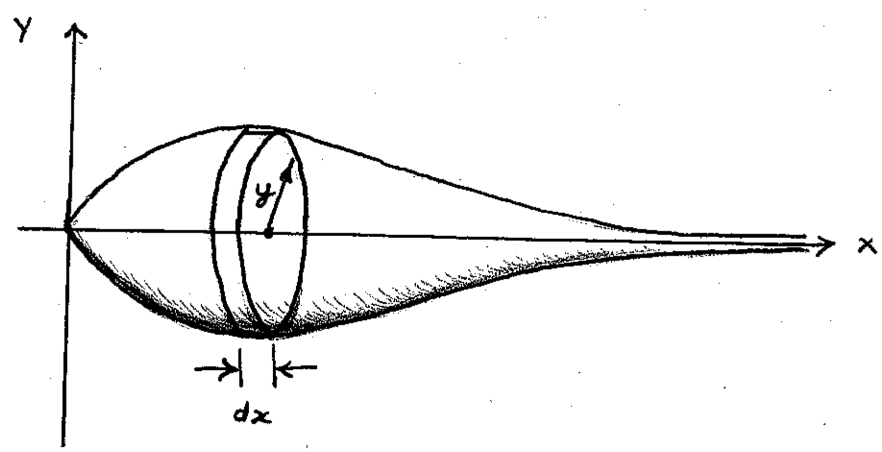
2. Consider the function $y = \frac{\sqrt{x^3}}{1+x^4}$ for $0 \le x < \infty$. Let R be the region bounded by:

- The line $x = 0$, the line $y = 0$, and the curve $y = \dfrac{\sqrt{x^3}}{1+x^4}$.

Find the **exact** volume of the solid that is generated when the region R is revolved around the x-axis.

The volume created when R is revolved around the x-axis looks something like the following:



The volume of the disk shown above is:

$$\text{Disk volume} = \pi \cdot y^2 \cdot dx$$

$$= \pi \cdot \left(\frac{\sqrt{x^3}}{1 + x^4} \right)^2 dx$$

The volume of the entire shape is:

$$\text{Volume} = \int_0^\infty \pi \cdot \frac{x^3}{(1+x^4)^2}\, dx$$

$$= \lim_{a \to \infty} \int_0^a \pi \cdot \frac{x^3}{(1+x^4)^2}\, dx$$

$$= \lim_{a \to \infty} \left[-\frac{\pi}{4} \cdot (1 + x^4)^{-1} \right]_0^a$$

$$= \lim_{a \to \infty} \frac{-\pi}{4(1+a^4)} + \frac{\pi}{4} = \frac{\pi}{4}$$

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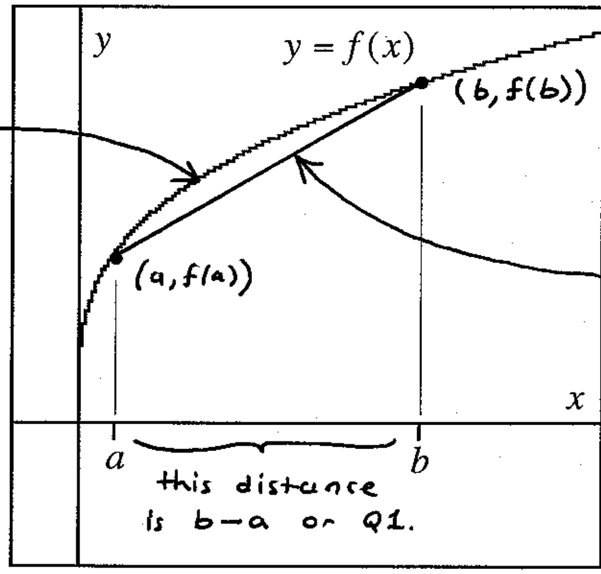
3. The following graph shows the function $f(x)$ and the points $x = a$ and $x = b$ $(a < b)$. Three quantities are defined using this graph as follows:

QII is the arc length of this piece of the graph.



The length of this line is $\sqrt{(b-a)^2 + (f(b)-f(a))^2}$ or QIII.

Quantity I: $\text{QI} = b - a$

Quantity II: $\text{QII} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Quantity III: $\text{QIII} = \sqrt{(b-a)^2 + (f(b) - f(a))^2}$

Which ONE of the following MUST be true?

(i) QI < QII < QIII

(ii) QIII < QI < QII

(iii) QII < QI < QIII

(iv) QI < QIII < QII

(iv) Must be true.

(v) QII < QIII < QI

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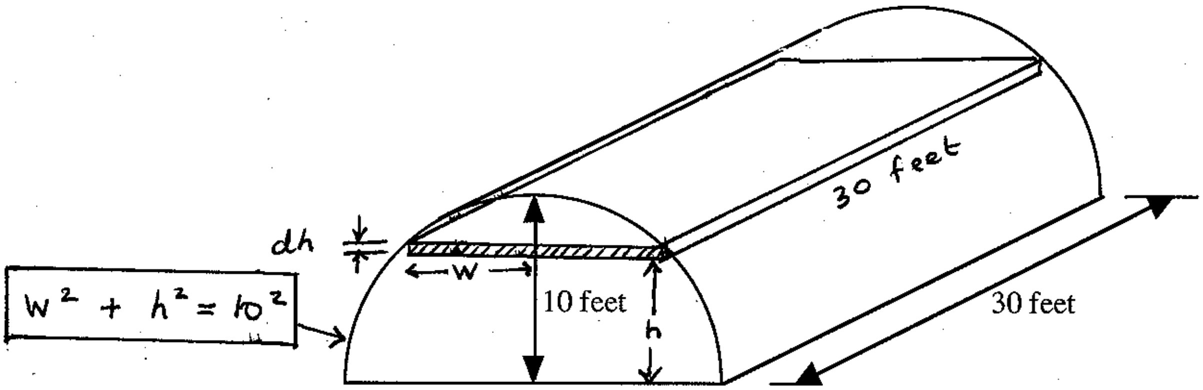
4. The storage shed shown in the diagram below is in the shape of a half cylinder. The radius of the cylinder is 10 feet and the length of the shed is 30 feet. Suppose the shed is completely filled with sawdust. The density of the sawdust at a height of h feet above the floor is equal to $(20 - h)$ pounds per cubic foot. If sawdust is extracted through a vent in the top of the roof, calculate the amount of work that must be done to clear the shed of sawdust when it is completely full.



Work = (Force) · (Distance moved).

The distance moved is $10 - h$ feet.

The force exerted by gravity on the slice shown is:

$$\text{Force} = (\text{Density})(\text{Volume})$$
$$= (20-h) \cdot 2w \cdot (30) \cdot dh$$
$$= (20-h) \cdot 2\sqrt{10^2 - h^2} \cdot (30) \cdot dh$$

The work done to remove the slice of sawdust shown is:

$$\text{Work} = (10-h)(20-h) \cdot 2 \cdot \sqrt{10^2 - h^2} \cdot (30) \cdot dh$$

The work needed to clear the entire shed is:

$$\text{Total work} = \int_0^{10} (10-h)(20-h) \cdot 2 \cdot \sqrt{10^2 - h^2} \cdot (30) \cdot dh$$

$$= 460287.52 \text{ foot-pounds.}$$

<u>NOTE:</u> Since the units of density were <u>pounds</u> per cubic foot we did not have to multiply by $g = 9.8\ m/s^2$ to get force. This is because pounds are already a unit of force.

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5. Find solutions to the differential equations below, subject to the given initial conditions. In each case, *demonstrate that your answer is correct* by providing step-by-step work to show how your answer was obtained.

(a) $$\frac{dP}{dt} = 4P - 8 \qquad P(0) = 20.$$

$$\frac{dP}{dt} = 4(P-2)$$

$$\int \frac{1}{P-2}\, dP = \int 4\, dt$$

$$\ln(|P-2|) = 4t + C$$

$$P = 2 + Ae^{4t}$$

where $A = \pm e^{C}$

To determine the value of A, use $P(0) = 20$.

$$20 = 2 + A$$

$$18 = A.$$

The final answer is:

$$P(t) = 2 + 18e^{4t}$$

(b) $$\frac{dy}{dx} + \underbrace{x^2}_{p(x).} \cdot y = e^{\frac{-1}{3}x^3} \qquad y(0) = 1.$$

It is not possible to separate the variables so we will use integrating factors instead.

$p(x) = x^2$ so $I = e^{\int p(x)\,dx} = e^{\frac{1}{3}x^3}$.

$$e^{\frac{1}{3}x^3} \cdot \frac{dy}{dx} + x^2 \cdot e^{\frac{1}{3}x^3} \cdot y = e^{\frac{1}{3}x^3} \cdot e^{-\frac{1}{3}x^3}$$

$$\frac{d}{dx}\left(e^{\frac{1}{3}x^3} \cdot y\right) = 1$$

Integrating both sides:

$$e^{\frac{1}{3}x^3} \cdot y = x + C$$

So:

$$y = \frac{x + C}{e^{\frac{1}{3}x^3}}$$

To determine C use $y(0) = 1$.

$$1 = \frac{C}{1} \quad \text{so } C = 1$$

and the final answer is:

$$y(x) = \frac{x + 1}{e^{\frac{1}{3}x^3}}$$

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6. Find the formula for $y(x)$, the solution of initial value problem given below. Your final answer should contain no unspecified constants.

$$y'' - 5y' + 6y = e^x \qquad y(0) = 0 \qquad y'(0) = 1.$$

Step 1: Solve the homogeneous equation.

$$y'' - 5y' + 6y = 0$$

Char. Equation: $r^2 - 5r + 6 = 0$

$$(r-2)(r-3) = 0$$

The roots of the characteristic equation are $r=2$ and $r=3$.

Homogeneous solution: $y_h(x) = C_1 e^{2x} + C_2 e^{3x}$.

Step 2: Create the Particular Solution

Function & derivatives	What's important
$N(x) = e^x$ $N'(x) = e^x$	e^x

Particular solution is: $y_p(x) = F \cdot e^x$ $\quad F =$ constant.

To determine F, plug $y_p(x)$ into the nonhomogeneous D.E.

$$y_p''(x) - 5y_p'(x) + 6y_p(x) = e^x$$

$$F \cdot e^x - 5F \cdot e^x + 6 \cdot F \cdot e^x = e^x$$

$$2Fe^x = e^x \quad \text{so} \quad F = 1/2.$$

Step 3: Use Initial Conditions.

$$y(x) = y_h(x) + y_p(x) = C_1 e^{2x} + C_2 e^{3x} + \tfrac{1}{2}e^x.$$

$y(0) = 0$: $\quad C_1 + C_2 + \frac{1}{2} = 0$

$y'(0) = 1$: $\quad 2C_1 + 3C_2 + \frac{1}{2} = 1$

Solving these for C_1 and C_2 gives: $C_1 = -2 \quad C_2 = 3/2$

and the final answer is:

$$y(x) = -2e^{2x} + \frac{3}{2}e^{3x} + \frac{1}{2}e^x.$$