Outline

1. Method of Undetermined Coefficients.
   (a) Sum Rule.
   (b) Modification Rule.

Test #2: This Friday during lecture time.

1. Method of Undetermined Coefficients.

- Good technique for solving to find $y(x)$ given something like:

$$y'' + 2y' + 5y = \underbrace{e^{0.5x} + 40\cos(10x) - 190\sin(10x)}_{N(x)}.$$

- Go through 3 steps:

① Solve homogeneous D.E. to find $y_h(x)$

② Use $N(x), N'(x), N''(x), \dots$ etc. to construct a particular solution $y_p(x)$.

③ $y(x) = y_h(x) + y_p(x)$ and use initial values to find constants.

(a) The Sum Rule

- Helps you to construct $y_p(x)$ when $N(x)$ has several different terms.

- Idea is to create a particular solution for each term in $N(x)$ and $y_p(x)$ is the sum of these.

Example

Find formula for $y(x)$ given:

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos(10x) - 190\sin(10x)$$

$$y(0) = \frac{1}{6.25} \qquad y'(0) = 40 + \frac{0.5}{6.25}$$

Solution

Step 1: Solve the Homogeneous D.E.

$$y'' + 2y' + 5y = 0$$

Char Eqn: $r^2 + 2r + 5 = 0$

Roots: $r = -1 + 2i \quad r = -1 - 2i$

$$\alpha = -1 \qquad \beta = 2$$

The formula $y_h(x)$ is:

$$y_h(x) = C_1 e^{\alpha x} \cdot \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$

$$\boxed{y_h(x) = C_1 e^{-x} \cdot \cos(2x) + C_2 e^{-x} \cdot \sin(2x)}$$

Step 2: Create the Particular Solution.

- Sum rule helps us here.

$$N(x) = e^{0.5x} + 40\cos(10x) - 190\sin(10x)$$

- First, pretend that $N(x) = e^{0.5x}$ and find a particular solution, $y_{p1}(x)$.

Function & Derivs	What's important
$N(x) = e^{0.5x}$, $N'(x) = 0.5e^{0.5x}$	$e^{0.5x}$

- For particular solution, use:

$$y_{p1}(x) = \underbrace{F}_{\text{constant.}} \cdot e^{0.5x}$$

- Determine value of F by plugging $y_{p1}(x)$ into nonhomogeneouse D.E. still pretending $N(x) = e^{0.5x}$.

$$y_{p1}''(x) + 2y_{p1}'(x) + 5y_{p1}(x) = e^{0.5x}$$

$$\left(0.25e^{0.5x} + e^{0.5x} + 5e^{0.5x}\right)F = e^{0.5x}$$

$$6.25\, e^{0.5x} \cdot F = e^{0.5x}$$

$$F = \frac{1}{6.25}.$$

<u>So:</u>

$$\boxed{y_{p1}(x) = \frac{1}{6.25} e^{0.5x}}$$

- Next, find a particular solution pretending that $N(x) = 40 \cdot \cos(10x)$, called $y_{p2}(x)$

Function & Derivs	What's Important
$N(x) = 40\cos(10x)$	$\cos(10x)$
$N'(x) = -400\sin(10x)$	and
$N''(x) = -4000\cos(10x)$	$\sin(10x)$.

So: $y_{p2}(x) = F \cdot \cos(10x) + G \cdot \sin(10x)$

contants

- To find F and G, plug $y_{p2}(x)$ into the nonhomogeneous D.E. pretending $N(x) = 40\cos(10x)$.

$$y_{p2}''(x) + 2y_{p2}'(x) + 5y_{p2}(x) = 40\cos(10x)$$

$$-100F \cdot \cos(10x) - 100G\sin(10x)$$
$$-20F \cdot \sin(10x) + 20G\cos(10x)$$
$$+5 \cdot F \cdot \cos(10x) + 5 \cdot G\sin(10x)$$
$$= 40 \cdot \cos(10x)$$

Equate coefficients of $\sin(10x)$ and $\cos(10x)$:

<u>$\sin(10x)$:</u> $-100G - 20F + 5G = 0$

$$-95G - 20F = 0 \quad \ldots ①$$

<u>$\cos(10x)$:</u> $-100F + 20G + 5F = 40$

$$-95F + 20G = 40 \quad \ldots ②$$

$$-475F + 100G = 200$$

$$-\frac{2000}{95}F - 100G = 0$$

$$\left(-475 - \frac{2000}{95}\right)F = 200$$

$$F = \frac{200}{-475 - \frac{2000}{95}}$$

$$G = \frac{4000}{-475(95) - 2000}.$$

So second particular solution is:

$$\boxed{y_{p2}(x) = \frac{200}{-475 - \frac{2000}{95}} \cdot \cos(10x) + \frac{4000}{-475(95) - 2000} \cdot \sin(10x)}$$

- Finally, pretend that $N(x) = -190 \sin(10x)$ and find a particular solution $y_{p3}(x)$.

Get:

$$\boxed{y_{p3}(x) = \frac{-200}{-475 - \frac{2000}{95}} \cos(10x) + \left(2 - \frac{4000}{-475 - 2000}\right) \cdot \sin(10x)}$$

$\underline{y(0)=0:}$

$$\frac{1}{6.25} = c_1 e^{-0}\cdot\cos(0) + c_2 e^{-0}\cdot\sin(0) + \frac{1}{6.25}e^{(0.5)(0)} + 2\cdot\sin(0)$$

$$\frac{1}{6.25} = c_1 + \frac{1}{6.25}$$

$\underline{\text{so}}$ $\boxed{c_1 = 0.}$

$\underline{y'(0) = 40 + \frac{0.5}{6.25}:}$

$$40 + \frac{0.5}{6.25} = -\overset{=0}{\cancel{c_1}}e^{-0}\cdot\cos(0) - \overset{=0}{\cancel{c_1}}e^{-0}\cdot\sin(0)\cdot 2 + -c_2 e^{-0}\cdot\sin(0) + c_2 e^{-0}\cdot\cos(0)\cdot 2 + \frac{0.5}{6.25}e^{0} + 2\cdot 10\cdot\cos(0)$$

- The overall particular solution for this problem is:

$$\boxed{\begin{aligned} y_p(x) &= y_{p1}(x) + y_{p2}(x) + y_{p3}(x) \\ &= \frac{1}{6.25} e^{0.5x} + 2 \cdot \sin(10x) \end{aligned}}$$

Step 3: Use Initial Conditions.

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1 e^{-x} \cos(2x) + C_2 e^{-x} \sin(2x) \\ &\quad + \frac{1}{6.25} e^{0.5x} + 2 \cdot \sin(10x). \end{aligned}$$

- Use $y(0) = \frac{1}{6.25}$ $y'(0) = 40 + \frac{0.5}{6.25}$ to determine C_1 and C_2.

$$40 + \frac{0.5}{6.25} = 2C_2 + \frac{0.5}{6.25} + 20$$

$$\boxed{C_2 = 10}$$

- Putting all of this together:

Final answer:

$$\boxed{y(x) = 10e^{-x} \cdot \sin(2x) + \frac{1}{6.25}e^{0.5x} + 2 \cdot \sin(10x).}$$