Outline

1. Integrating factors.

2. Homogeneous, second order differential equations with constant coefficients.

Final exam: [redacted]. Friday 12/12/08
8:30AM – 11:30AM.

1. Integrating Factors

- Good way to solve D.E.'s that look like:

$$\frac{dy}{dx} + p(x) \cdot y = r(x)$$

to find a formula for $y = y(x)$.

- $I = e^{\int p(x)\,dx}$.
- Multiply everything in the D.E. by I.
- Reverse product rule.

Example

Find a formula for $y(x)$:

$$\frac{dy}{dx} + \frac{x}{\sqrt{x^2+1}}\, y = x \qquad y(0) = 1.$$

Solution

$$p(x) = \frac{x}{\sqrt{x^2+1}}$$

$$\int p(x)\,dx = \int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}$$

OK to ignore +C here

$$I = e^{\int p(x)\,dx} = e^{\sqrt{x^2+1}}$$

$$\underbrace{e^{\sqrt{x^2+1}} \cdot \frac{dy}{dx} + e^{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2+1}} \cdot y} = x \cdot e^{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(e^{\sqrt{x^2+1}} \cdot y\right) = x \cdot e^{\sqrt{x^2+1}}$$

$$\int \frac{d}{dx}\left(e^{\sqrt{x^2+1}} \cdot y\right) dx = \int x \cdot e^{\sqrt{x^2+1}}\,dx$$

$$e^{\sqrt{x^2+1}} \cdot y = \sqrt{x^2+1} \cdot e^{\sqrt{x^2+1}} - e^{\sqrt{x^2+1}} + C$$

<u>not</u> OK to ignore $+C$ here

Determine C using $y(0)=1$:

$$e^1 \cdot 1 = 1 \cdot e^1 - e^1 + C$$

$$C = e^1$$

Use $C = e^1$ and make y the subject:

$$y = y(x) = \frac{\sqrt{x^2+1} \cdot e^{\sqrt{x^2+1}} - e^{\sqrt{x^2+1}} + e^1}{e^{\sqrt{x^2+1}}}$$

2. Homogeneous, Second Order Differential Equations with Constant Coefficients

second order: $y''(t)$ present

$$a \cdot \frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y = 0$$

a, b, c numbers

Homogeneous means sum of derivatives + function = 0

Goal: Find a formula for $y(t)$.

To do this, create a <u>characteristic equation</u>.

D.E. $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = 0$

<u>Char. Eqn.</u> $a \cdot r^2 + b \cdot r + c = 0$.

- The roots of the characteristic equation tell you how to create $y(t)$.

<u>Case 1: Characteristic Equation has two distinct real roots.</u>

e.g. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0$

Find $y(t)$.

Solution

Char. Eqn.

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

Roots: $r = -1$ and $r = -2$.

Formula for $y(t)$:

$$y(t) = c_1 \cdot e^{-1 \cdot t} + c_2 \cdot e^{-2 \cdot t}$$

Roots of characteristic equation

Constants c_1 and c_2

Case 2: Characteristic Equation has a real, repeated root.

e.g. $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

Find $y(t)$.

Solution

Char. Eqn.: $r^2 + 4r + 4 = 0$

$$(r+2)(r+2) = 0$$

Repeated root: $r = -2$.

Formula for $y(t)$

$$y(t) = (C_1 + C_2 t) \cdot e^{-2 \cdot t}$$

Factor of 't'

Case 3: Characteristic Equation has Complex Roots.

Char. Eqn: $a \cdot r^2 + b \cdot r + c = 0$

Get complex roots when: $b^2 - 4ac < 0$.

The roots are given by:

$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$

where: $\alpha = -b/2a$

$\beta = \sqrt{4ac - b^2}/2a$

$i = \sqrt{-1}$.

$$\boxed{\text{The formula for } y(t) \text{ is: } \quad y(t) = C_1 e^{\alpha t} \cdot \cos(\beta t) + C_2 e^{\alpha t} \cdot \sin(\beta t)}$$

Example

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0$$

Find a formula for $y(t)$.

Solution

Char. Eqn.: $r^2 + 2r + 2 = 0$

$a=1 \qquad b=2 \qquad c=2$

$$b^2 - 4ac = 4 - 4(1)(2) = -4 < 0$$

So we'll have complex roots.

$\alpha = -b/2a \qquad\qquad \beta = \sqrt{4ac-b^2}/2a$

$= -2/2 \qquad\qquad = 2/2$

$= -1 \qquad\qquad = 1$

Formula for $y(t)$:

$$y(t) = C_1 e^{-1\cdot t} \cdot \cos(1\cdot t) + C_2 e^{-1\cdot t} \cdot \sin(1\cdot t)$$