Outline

1. Logistic Equation.
2. Setting up differential equations.
3. Integrating factors.

Final Exam:
Friday December 12
8:30 am – 11:30 am.

1. Logistic Equation

$P(t) =$ # people infected.

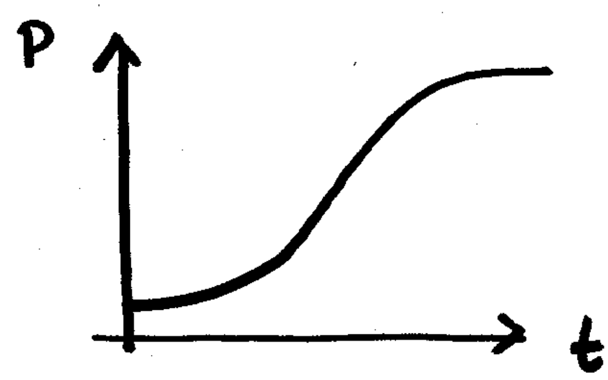
$t =$ time (minutes).

$$\frac{dP}{dt} = -0.0119 \cdot P^2 + 0.348 \cdot P$$

$P(0) = 1.$

Solve to find a formula for $P(t)$.

Check:



Looking for $P(t)$ with this shape.

Solution

① Separate Variables.

$$\frac{dP}{dt} = -0.0119 \cdot \underbrace{P \cdot (P - 29.24)}_{\text{divide}}$$

$$\frac{1}{P \cdot (P - 29.24)} \, dP = -0.0119 \cdot dt$$

② Integrate.

$$\int \frac{1}{P(P-29.24)} \, dP = \int -0.0119 \, dt$$

Partial fractions:

$$\frac{1}{P(P-29.24)} = \frac{A}{P} + \frac{B}{P-29.24}$$

Equating coefficients of P:

$\underline{P^1:}$ $A + B = 0$

$\underline{P^0:}$ $-29.24\,A = 1$

$$A = \frac{-1}{29.24} \qquad B = \frac{1}{29.24}.$$

$$\int \frac{1}{P(P-29.24)}\,dP = \int \frac{\frac{-1}{29.24}}{P}\,dP + \int \frac{\frac{1}{29.24}}{P-29.24}\,dP$$

$$= \frac{-1}{29.24}\ln(|P|) + \frac{1}{29.24}\ln(|P-29.24|)$$

$$= \frac{1}{29.24}\ln\left(\left|\frac{P-29.24}{P}\right|\right)$$

Putting integrals together:

$$\frac{1}{29.24} \ln\left(\left|\frac{P-29.24}{P}\right|\right) = -0.0119t + C$$

③ Make P the subject.

$$\ln\left(\left|\frac{P-29.24}{P}\right|\right) = -0.348t + C$$

$$e^{\ln\left(\left|\frac{P-29.24}{P}\right|\right)} = e^{-0.348t + C}$$

$$\left|\frac{P-29.24}{P}\right| = e^{c} \cdot e^{-0.348t}$$

$$\frac{P-29.24}{P} = A \cdot e^{-0.348t}$$

$$A = \pm e^{c}$$

$$P - 29.24 = A \cdot e^{-0.348t} \cdot P$$

$$P - P \cdot A \cdot e^{-0.348t} = 29.24$$

$$P \cdot \left[1 - A \cdot e^{-0.348t}\right] = 29.24$$

$$P = \frac{29.24}{1 - Ae^{-0.348t}}$$

④ Evaluate A.

When $t = 0$ $P = 1$.

$$1 = \frac{29.24}{1 - A}$$

$$1 - A = 29.24$$

$$A = -28.24.$$

Final answer:

$$P = P(t) = \frac{29.24}{1 + 28.24\, e^{-0.348t}}$$

2. Setting up Differential Equations.

- **Prototype:**

$$\boxed{\text{Derivative} = \underset{\text{in}}{\text{Rate}} - \underset{\text{out.}}{\text{Rate}}}$$

Example

A tank holds 1000 gallons of water with 100 lbs of salt initially dissolved in it.

Brine is pumped into the tank at a rate of 10 gallons per minute. Each gallon of brine carries 2 lb. of salt.

The brine mixes with the solution in the tank, and the mixture is pumped out at 10 gallons/minute.

Find an initial value and differential equation for $A(t)$, the amount of salt in the tank in lbs.

<u>Solution:</u>

Initial value: $A(0) = 100.$

Differential equation:

$$\text{Derivative} = \frac{\text{Change in salt}}{\text{Change in time}} \quad \frac{\text{lbs.}}{\text{minute}}$$

$$= \frac{dA}{dt} \quad \text{lbs/minute}$$

Rate in:

- Brine enters 10 gal/minute
- 2 lbs of salt per gallon of brine.

$$\text{Rate in} = (10)(2) \quad \frac{\text{gal}}{\text{min}} \cdot \frac{\text{lb}}{\text{gal}} = \frac{\text{lb}}{\text{minute.}}$$

Rate out:

- Mixture leaves 10 gal/minute.
- Concentration of Salt $= \dfrac{A(t)}{1000}$ lb/gal.

Rate out $= \dfrac{A(t)}{1000} \cdot 10 \quad \dfrac{\text{lb}}{\text{minutes}}$.

Differential equation:

$$\frac{dA}{dt} = 20 - \frac{A(t)}{1000} \cdot 10$$

3. Integrating Factors

- Good method for solving D.E.'s that can be put into the form:

$$\boxed{\frac{dy}{dx} + p(x) \cdot y = r(x).}$$

- Integrating factor: $I = e^{\int p(x)\,dx}$

Example

Solve: $$\frac{dy}{dx} + \underbrace{\tan(x)}_{p(x)} \cdot y = \sin(2x)$$

to find $y(x)$.

Solution

$$\int p(x)\,dx = \int \tan(x)\,dx$$

$$= \ln(|\sec(x)|). + \overset{\text{ignore}}{C}$$

Integrating factor:

$$I = e^{\int p(x)dx} = e^{\ln(|\sec(x)|)}$$

$$= \sec(x)$$

Multiply D.E. by I.

$$\sec(x)\frac{dy}{dx} + \sec(x)\tan(x)\cdot y = \sec(x)\cdot\sin(2x)$$

$$\frac{d}{dx}\left(\sec(x)\cdot y\right) = 2\cdot\sin(x)$$

$$\int \frac{d}{dx}\left(\sec(x)\cdot y\right)dx = \int 2\cdot\sin(x)\cdot dx$$

$$\sec(x)\cdot y = -2\cdot\cos(x) + C$$

$$y = \frac{-2\cdot\cos(x) + C}{\sec(x)}$$

$$\sin(2x) = 2\cdot\sin(x)\cos(x)$$

$$\sec(x)\cdot\sin(2x) = \frac{1}{\cos(x)}\cdot 2\cdot\sin(x)\cdot\cos(x)$$

$$= 2\cdot\sin(x)$$