Outline

1. Improper integrals.
2. Area between curves.
3. Volumes of Revolution.

1. Improper Integrals

$$\int_3^\infty \frac{1}{x^2}\,dx \qquad \int_{-1}^0 \frac{1}{\sqrt{x+1}}\,dx$$

$$\int_{-\infty}^{-1} \frac{1}{x^2+2}\,dx \qquad \int_1^2 \frac{1}{x^2-4}\,dx$$

Type I

One limit of integration is infinite.

Type II

Vertical asymptote at endpoint.

- Today we will look at:

Type III $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Both limits of integration are infinite.

Type IV $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Vertical asymptote occurs between limits of integration.

Type III

- Both limits of integration are infinite.
- Approach is to:

1. Split $\int_{-\infty}^{\infty} f(x)\,dx$ at some finite $x = x_0$ into $\int_{x_0}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{x_0} f(x)\,dx$.

② Determine convergence/ divergence of $\int_{x_0}^{\infty} f(x)dx$ and $\int_{-\infty}^{x_0} f(x)dx$ separately.

③ If $\int_{x_0}^{\infty} f(x)dx$ <u>and</u> $\int_{-\infty}^{x_0} f(x)dx$ <u>both</u> converge then $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

④ If either of $\int_{x_0}^{\infty} f(x)dx$ or $\int_{-\infty}^{x_0} f(x)dx$ (or both) diverge then $\int_{-\infty}^{\infty} f(x)dx$ diverges.

Example

Does $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converge or diverge?

Solution

Determine whether or not both $\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$ and $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$ both converge.

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \int_{0}^{a} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to\infty} \Big[\arctan(x)\Big]_{0}^{a}$$

$$= \lim_{a \to \infty} \arctan(a)$$

$$= \pi/2$$

Next:

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \Big[\arctan(x)\Big]_a^0$$

$$= \lim_{a \to -\infty} -\arctan(a)$$

$$= \pi/2$$

So $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges

to $\frac{\pi}{2} + \frac{\pi}{2} = \pi$.

Example

Does $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$ converge or diverge?

Solution

Might think that since $f(x) = \frac{x}{1+x^2}$ is an odd function and for any finite number a,

$$\int_{-a}^{a} f(x)\, dx = 0$$

that $\int_{-\infty}^{\infty} f(x)\, dx = 0$ and the integral converges.

$(-7, 7)$ Symmetric interval

$(-\infty, \infty)$ Not symmetric.

But <u>NOT OK</u> to reason this way when $\pm\infty$ involved.

To determine convergence/ divergence;

$$\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{a\to\infty} \int_0^a \frac{x}{1+x^2}\,dx$$

$$= \lim_{a\to\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_0^a$$

$$= \lim_{a\to\infty} \frac{1}{2}\ln(1+a^2) - \frac{1}{2}\ln(1)$$

$$= +\infty$$

So $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ diverges.

Type IV

- Function being integrated has a vertical asymptote between the limits of integration.
- To handle these:

① Split $\int_{\alpha}^{\beta} g(x)\,dx$ into two type II integrals at the point $(x = x_0)$ where the vertical asymptote occurs.

② Determine convergence/divergence of $\int_{x_0}^{\beta} g(x)\,dx$ and $\int_{\alpha}^{x_0} g(x)\,dx$ separately.

③ If both $\int_{x_0}^{\beta} g(x)\,dx$ and $\int_{\alpha}^{x_0} g(x)\,dx$ both converge, then $\int_{\alpha}^{\beta} g(x)\,dx$ converges.

<u>Example</u>

Does $\int_{-1}^{1} \frac{1}{x^3}\,dx$ converge or diverge?

Type IV Improper Integral

- Vertical asymptote at a point between the limits of integration.

e.g. $\int_{-1}^{3} \frac{1}{x^2}\,dx$

vertical asymptote at $x = 0$.

- Split integral at the vertical asymptote into a pair of Type II improper integrals.

- Both Type II integrals have to converge in order for the Type IV integral to converge.

Example

Does $\int_{-2}^{2} \frac{1}{x^3}\,dx$ converge or diverge?

Solution

Investigate convergence/divergence of:

$\int_0^2 \frac{1}{x^3}\,dx$ and $\int_{-2}^{0} \frac{1}{x^3}\,dx$.

Both need to converge for

$\int_{-2}^{2} \frac{1}{x^3}\,dx$ to converge.

$$\int_0^2 \frac{1}{x^3}\,dx = \lim_{b\to 0^+} \int_b^2 \frac{1}{x^3}\,dx$$

$$= \lim_{b\to 0^+} \left[\frac{x^{-2}}{-2}\right]_b^2$$

$$= \lim_{b\to 0^+} \frac{1}{2b^2} - \frac{1}{8}$$

$$= +\infty.$$

So $\int_0^2 \frac{1}{x^3}\,dx$ and $\int_{-2}^{2} \frac{1}{x^3}\,dx$ diverge.