Outline

1. Type II Improper Integrals.
2. Review questions.

Office hours:

Today noon - 2pm
6124 Wean Hall.

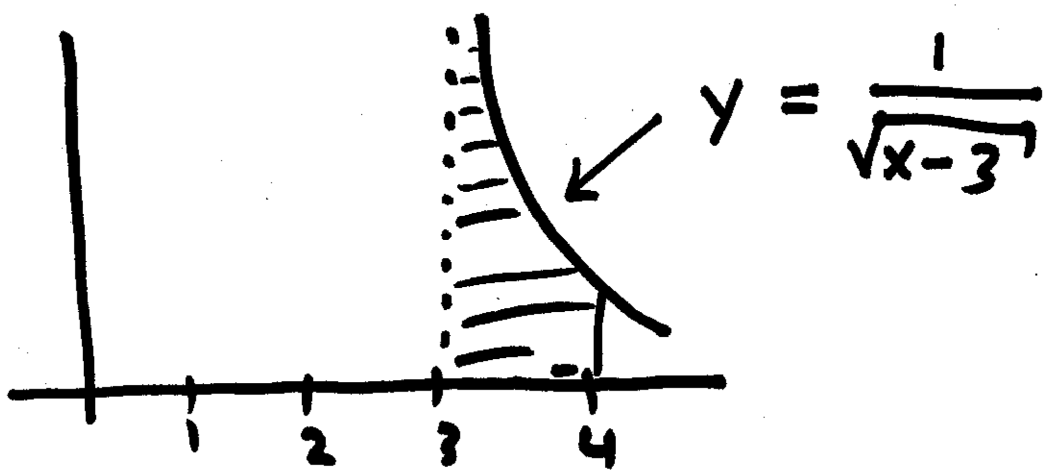
1. Type II Improper Integrals

- Function $f(x)$ has a vertical asymptote at some point in the interval $[a,b]$.
- Main question: Is $\int_a^b f(x)\,dx$ <u>finite</u> (integral converges) or is $\int_a^b f(x)\,dx$ <u>infinite</u> (integral diverges).

Example

Does $\int_3^4 \frac{1}{\sqrt{x-3}}\,dx$ converge or diverge? If convergent, to what?

Solution



$$\int_3^4 \frac{1}{\sqrt{x-3}}\,dx = \lim_{b \to 3^+} \int_b^4 \frac{1}{\sqrt{x-3}}\,dx$$

$$= \lim_{b \to 3^+} \left[2\sqrt{x-3} \right]_b^4$$

$$= \lim_{b \to 3^+} 2\sqrt{4-3} - \underbrace{2\sqrt{b-3}}_{\substack{\to 0 \\ b \to 3^+}}$$

$$= 2 - 0$$

$$= 2$$

$\int_3^4 \frac{1}{\sqrt{x-3}}\,dx$ converges to 2.

First question from students:

$$\int \left(1 - \sin(\theta)\right)^2 \, d\theta$$

$$= \int 1 - 2\sin(\theta) + \underbrace{\sin^2(\theta)}_{\frac{1}{2}(1-\cos(2\theta))} \, d\theta$$

$$= \int 1 - 2\sin(\theta) + \frac{1}{2} - \frac{1}{2}\cos(2\theta) \, d\theta$$

$$= \frac{3}{2}\theta + 2\cos(\theta) - \frac{1}{4}\sin(2\theta) + C$$

$$\int_1^\infty x \cdot e^{-x}\, dx$$ Second question from students

$$\int_1^\infty x \cdot e^{-x} \cdot dx = \lim_{a\to\infty} \int_1^a x \cdot e^{-x} dx$$

Find antiderivative:

$$\int x \cdot e^{-x} dx \quad = -xe^{-x} + \int e^{-x} dx$$

$u = x$ $\qquad v' = e^{-x}$

$u' = 1$ $\qquad v = -e^{-x}$

$$= -xe^{-x} - e^{-x} + C.$$

$$\int_1^\infty x e^{-x} dx = \lim_{a\to\infty} \Big[-xe^{-x} - e^{-x}\Big]_1^a$$

$$= \lim_{a\to\infty} -ae^{-a} - e^{-a} + e^{-1} + e^{-1}$$

OK to assume: $\lim_{a\to\infty} a^n \cdot e^{-a} = 0$

$$\int_1^\infty xe^{-x}dx = 0 + 2e^{-1}$$

Third question from students

$\int_1^\infty x e^{-x} dx$ by comparison.

<u>Guess:</u> $\int_1^\infty xe^{-x} dx$ converges.

<u>Need:</u> Convergent integral

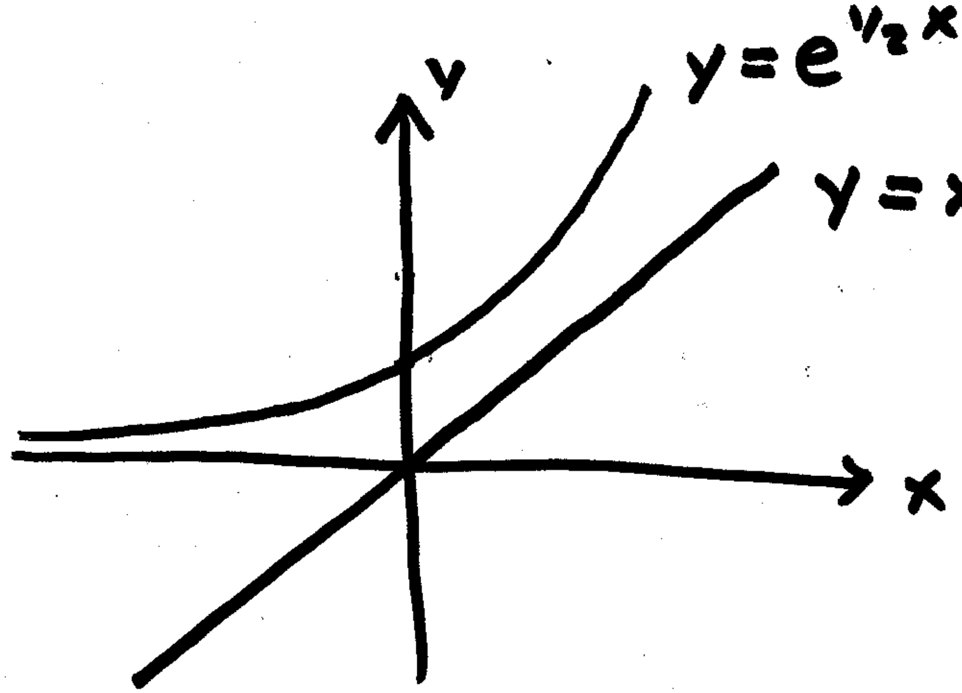
$$\int_1^\infty f(x)\,dx$$

which obeys: $x \cdot e^{-x} \leq f(x)$.

finding this is not easy.

when x is sufficiently large,

$$x \leq e^{\frac{1}{2}x}$$



$$x \leq e^{1/2 x}$$

$$x \cdot e^{-x} \leq e^{1/2 x} \cdot e^{-x}$$

$$x \cdot e^{-x} \leq e^{-1/2 x}$$

$$\int_c^\infty e^{kx}\,dx \to \begin{cases} \text{Diverges } k \geq 0 \\ \text{Converges } k < 0 \end{cases}$$

$\int_1^\infty e^{-1/2 x}\,dx$ converges. So

$\int_1^\infty x \cdot e^{-x}\,dx$ converges as well.

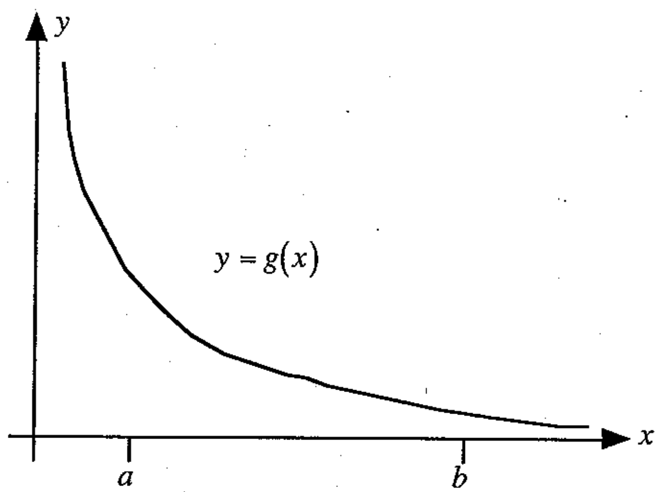
Handout 5: Review Problems for Lecture 10

The topics that will be covered on Unit Test 1 are:

- Integration formulas.
- U-substitution.
- Integration by parts.
- Integration using trigonometric formulas and identities.
- Trigonometric substitution.
- Partial fractions.
- Integration tricks such as polynomial long division and completing the square.
- Approximating integrals (Riemann sums, trapezoid and midpoint rules, Simpson's rule).
- Conditions under which different integration methods produce over and under estimates.
- Error estimates for trapezoid, midpoint and Simpson's rules.
- Improper integrals.

This (roughly) covers Section 5.5 and Chapter 6 of the textbook. The problems listed here are ones that involve important concepts and techniques from these sections of the book.

1. Consider the function $g(x)$ defined by the graph shown below.



Arrange the following quantities from smallest to largest:

- Left hand Riemann sum using n rectangles,
- Right hand Riemann sum using n rectangles,
- Midpoint rule using n rectangles,
- Trapezoid rule using n rectangles, and,
- $\int_a^b g(x)dx$.

2. Evaluate the following definite integral:

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

3. Evaluate the following definite integral:

$$\int_0^1 \frac{x}{x^2 + 4x + 13}\,dx\,.$$

4. In this problem you will be interested in approximating the value of the integral:

$$\int_1^2 e^{\frac{1}{x}} \cdot dx$$

(a) How many rectangles should you use to approximate the value of this integral using the Midpoint rule with an error of less than 0.01?

(b) Use the Midpoint rule and the number of rectangles calculated in Part (a) to approximate the value of the integral.

Answers

1. Right hand sum $\leq$ Midpoint $\leq$ Integral $\leq$ Trapezoid $\leq$ Left hand sum.

2. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \frac{40}{3}.$

3. This integral can be computed using a combination of u-substitution and completing the square.

$$\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - \int_0^1 \frac{1}{(x+2)^2+9}\,dx = \tfrac{1}{2}\cdot\ln\left(\tfrac{18}{13}\right) - \tfrac{\pi}{6} + \tfrac{2}{3}\cdot\arctan\left(\tfrac{2}{3}\right).$$

4.(a) Six rectangles should be used.

4.(b) Midpoint = 2.017420884.

Review Problem #2

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$\sqrt{16-x^2}$ Trig subst.

$$x = 4\cdot\sin(\theta)$$
$$dx = 4\cdot\cos(\theta)\,d\theta$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{64\sin^3(\theta)}{4\cos(\theta)}\,4\cos(\theta)\,d\theta$$

$$= 64\int \sin^3(\theta)\,d\theta$$

$$= 64\int \sin(\theta)\cdot\sin^2(\theta)\cdot d\theta$$

$$= 64\int \sin(\theta)\cdot\left(1-\cos^2(\theta)\right)d\theta$$

$$= 64\int \sin(\theta)\,d\theta$$

$$- 64\int \sin(\theta)\cdot\cos^2(\theta)\,d\theta$$

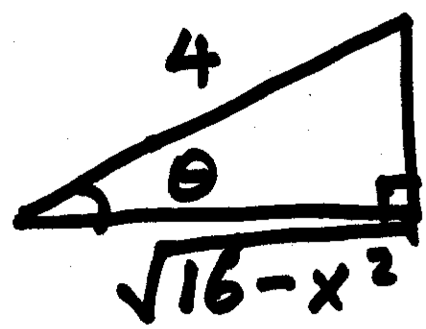
$u = \cos(\theta)$

$$= -64\cos(\theta) + \frac{64}{3}\cos^3(\theta) + C$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = -64\cos(\theta) + \frac{64}{3}\cos^3(\theta) + C$$

$$x = 4\cdot\sin(\theta)$$

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{x}{4}$$



so:

$$\cos(\theta) = \frac{\sqrt{16-x^2}}{4}$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = -16\sqrt{16-x^2} + \frac{1}{3}(16-x^2)^{3/2} + C$$

So:

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$$= -16\sqrt{16-12} + \frac{1}{3}(16-12)^{3/2}$$

$$- \left(-16\sqrt{16-0} + \frac{1}{3}(16-0)^{3/2}\right)$$

$$= 40/3.$$

Review Problem #3

$$\int \frac{x}{\underbrace{x^2+4x+13}_{\text{factor: } (x+2)^2+9}}\,dx$$

$(\quad 13)(\quad 1)$

$ax^2 + bx + c$

Discriminant : $\Delta = b^2 - 4ac$

$= 4^2 - 4(1)(13)$

$= -36$

$$\int \frac{x}{\underbrace{x^2+4x+13}_{\text{derivative} = 2x+4}}\,dx$$

$x+2 = \frac{1}{2}(2x+4)$

$$= \int \frac{x+2-2}{x^2+4x+13}\,dx$$

$$= \int \frac{x+2}{x^2+4x+13}\,dx - 2\int \frac{1}{x^2+4x+13}\,dx$$

$u = x^2+4x+13$

$du = (2x+4)\,dx$

$$= \frac{\ln(|x^2+4x+13|)}{2} - 2\int \frac{1}{(x+2)^2+9}\,dx$$

$u = x+2$

$a = 3$

$$= \frac{\ln(|x^2+4x+13|)}{2}$$

$$- \quad 2 \cdot \frac{1}{3} \cdot \tan^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int_0^1 \frac{x}{x^2+4x+13}\,dx = \frac{\ln(18)}{2} - \frac{2}{3}\tan^{-1}(1)$$

$$- \frac{\ln(13)}{2} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)$$