Solutions to Homework #8

Problems from Pages 427-429 (Section 8.2)

10. To decide whether or not the infinite series $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ converges or diverges we will use the n^{th} term test for divergence. For this infinite series $a_n = \frac{n+1}{2n-3}$. Taking the limit of this as $n \to \infty$ gives $\text{Lim}_{n \to \infty} a_n = 0.5 \neq 0$. Since the limit is not equal to zero, the infinite series $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges.

12. To decide whether or not the infinite series $\sum_{k=1}^{\infty} \frac{k \cdot (k+2)}{(k+3)^2} = \sum_{k=1}^{\infty} \frac{k^2+2k}{k^2+6k+9}$ converges or diverges we will use the n^{th} term test for divergence. For this infinite series $a_k = \frac{k^2+2k}{k^2+6k+9}$. Taking the limit of this as $k \to \infty$ gives $\text{Lim}_{k \to \infty} a_k$ $= 1 \neq 0$. Since the limit is not equal to zero, the infinite series $\sum_{k=1}^{\infty} \frac{k \cdot (k+2)}{(k+3)^2}$ diverges.

20. Notice that the term of the series can be broken down into a difference of two terms using the technique of **partial fractions.**

$$\frac{2}{n^2+4n+3} = \frac{2}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}.$$

Carrying out the computations for partial fractions gives $A = 1$ and $B = -1$. Writing out the first few partial sums gives the following results:

$$S_1 = \frac{1}{2} - \frac{1}{4}.$$

$$S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}.$$

$$S_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}.$$

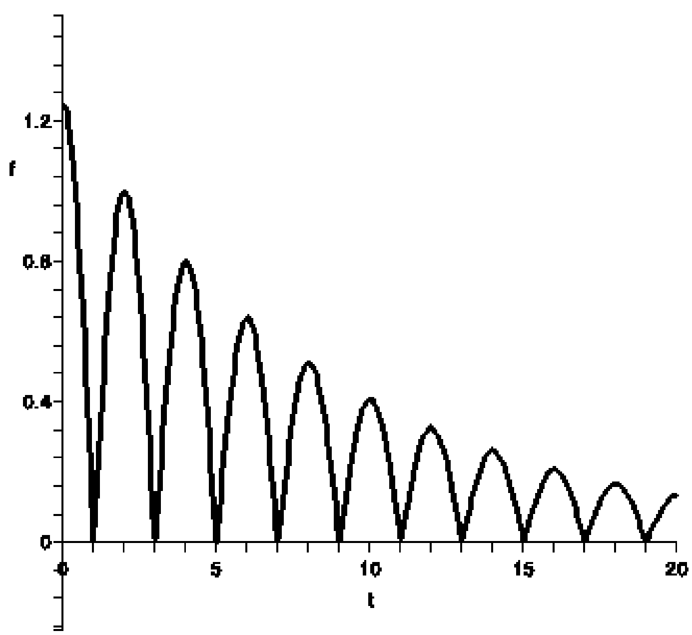
$$S_4 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7}.$$

The pattern that is developing is that each fraction that appears in the partial sum (with the exception of ½ and 1/3) is eventually canceled by a later fraction. This means that the limit of the partial sums as $N \to \infty$ will be:

$$\text{Lim}_{N \to \infty} S_N = ½ + 1/3 = 5/6.$$

Since the limit of the partial sums exists and is finite, the infinite series $\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3}$ converges. The sum of the series is 5/6.

34. A graph showing the height of the ball as it bounces is given below. The height of each bounce is indicated.



(a) The total distance traveled by the ball is given by the infinite series:

$$H + 2Hr + 2Hr^2 + 2Hr^3 + ... = H + \sum_{k=1}^{\infty} 2Hr^k \text{ meters.}$$

The part of the series represented using Σ notation above is an infinite geometric series with $a = 2Hr$ and $r = r$. Common sense dictates that $0 < r < 1$ because each bounce of a ball is lower than the preceding bounce, so this geometric series converges and the summation formula for an infinite geometric series can be used:

$$\text{Total distance covered} = H + \frac{2Hr}{1-r} = \frac{H - Hr + 2Hr}{1-r} = H \cdot \frac{1+r}{1-r} \text{ meters.}$$

(b) The total time, t_k, that the ball needs to travel (up or down) a distance of Hr^k can be obtained by rearranging the constant acceleration kinematic equation:

$$\tfrac{1}{2} g t_k^{\ 2} = Hr^k$$

$$t_k = \sqrt{\frac{2Hr^k}{g}} = \sqrt{\frac{2H}{g}} \cdot \left(\sqrt{r}\right)^k,$$

where $g = 9.8$ ms^{-2}. Again referring to the graph showing the height of the ball as it bounces, the total time taken by the ball's bounces will be:

$$t_0 + 2t_1 + 2t_2 + \ldots = t_0 + 2 \cdot \sum_{k=1}^{\infty} t_k = \sqrt{\frac{2H}{g}} + 2 \cdot \sum_{k=1}^{\infty} \sqrt{\frac{2H}{g}} \cdot \left(\sqrt{r}\right)^k.$$

(c) The ball will come to rest only when it has been able to bounce an infinite number of times. The total time needed for the ball to accomplish this feat will be the sum of the series found in Part (b). The part of the series expressed using Σ notation in Part (b) is an infinite geometric series with $a = 2 \cdot \sqrt{\frac{2H}{g}}$ and $"r" = \sqrt{r}$. Assuming again that the bouncing ball in this problem resembles a realistic bouncing ball with $0 < r < 1$, we can employ the summation formula for a geometric series and obtain the following expression for the total time:

$$\sqrt{\frac{2H}{g}} + 2 \cdot \sum_{k=1}^{\infty} \sqrt{\frac{2H}{g}} \cdot \left(\sqrt{r}\right)^k = \sqrt{\frac{2H}{g}} + \frac{2 \cdot \sqrt{\frac{2H}{g}}}{1-\sqrt{r}} = \sqrt{\frac{2H}{g}} \cdot \frac{1+\sqrt{r}}{1-\sqrt{r}}.$$

Problems from Pages 436-437 (Section 8.3)

8. We will use the Integral test to determine the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. To do this we will use the function $f(x) = \frac{1}{x^2+1}$ which is both positive and decreasing for $x > 0$. According to the integral test, the convergence or divergence of the infinite series will be the same as the convergence or divergence of the improper integral:

$$\int_{1}^{\infty} f(x) \cdot dx .$$

Evaluating the improper integral (and noting the use of limit notation when evaluating the improper integral) we get:

$$\int_{1}^{\infty} \frac{1}{1+x^2} \cdot dx = \lim_{a \to \infty} \int_{1}^{a} \frac{1}{1+x^2} \cdot dx = \lim_{a \to \infty} \left[\arctan(x)\right]_1^a = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4} .$$

As the improper integral is convergent, the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is also convergent.

12. Note that a p-series of the form $\sum_{n=1}^{\infty} \frac{C}{n^p}$ is convergent whenever $p > 1$. Therefore the two series:

$$\sum_{n=1}^{\infty} \frac{5}{n^4} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{4}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{4}{n \cdot \sqrt{n}}$$

are both convergent series. The sum of the two series is therefore convergent, so the infinite series $\sum_{n=1}^{\infty} \frac{5}{n^4} + \sum_{n=1}^{\infty} \frac{4}{n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{5}{n^4} + \frac{4}{n \cdot \sqrt{n}} \right)$ is convergent.

Problems from Pages 446-447 (Section 8.4)

20. We will use the Ratio Test to determine the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. Here $a_n = \frac{n^2}{2^n}$ so that:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \cdot \frac{(n+1)^2}{n^2} .$$

Taking the limit of the absolute value of this ratio as $n \to \infty$ gives a limit of ½. As the limit is less than one, the Ratio test gives that the infinite series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges absolutely.

24. We will use the Ratio Test to determine the convergence or divergence of the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^4}$. Here $a_n = (-1)^{n+1}\frac{2^n}{n^4}$ so that:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\cdot 2^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(-1)^{n+1}\cdot 2^n} = -2\cdot\frac{n^4}{(n+1)^4}.$$

Taking the limit of the absolute value of this ratio as $n \to \infty$ gives a limit of 2. As the limit is greater than one, the Ratio test gives that the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^4}$ is divergent.

28. We will use the Ratio Test to determine the convergence or divergence of the infinite series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 5^{n-1}}{(n+1)^2\cdot 4^{n+2}}$. Here $a_n = \frac{(-1)^{n+1}\cdot 5^{n-1}}{(n+1)^2\cdot 4^{n+2}}$ so that:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\cdot 5^n}{(n+2)^2\cdot 4^{n+3}}\cdot\frac{(n+1)^2\cdot 4^{n+2}}{(-1)^{n+1}\cdot 5^{n-1}} = \frac{-5}{4}\cdot\frac{(n+1)^2}{(n+2)^2}$$

Taking the limit of the absolute value of this ratio as $n \to \infty$ gives a limit of 5/4. As the limit is greater than one, the Ratio test gives that the infinite series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 5^{n-1}}{(n+1)^2\cdot 4^{n+2}}$ is divergent.

36. Convergence test are easiest to apply when the general term of the series has been expressed (i.e. the formula that appears in Σ notation). Our first task is to find an expression for the general term of the infinite series:

$$\frac{2}{5}+\frac{2\cdot 6}{5\cdot 8}+\frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}+\frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}+\ldots$$

Assigning values of n to terms of the series as indicated below,

$$\underbrace{\frac{2}{5}}_{n=1}+\underbrace{\frac{2\cdot 6}{5\cdot 8}}_{n=2}+\underbrace{\frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}}_{n=3}+\underbrace{\frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}}_{n=4}+\ldots$$

the n^{th} (or general) term of the series can be written down as:

$$a_n = \frac{2 \cdot 6 \cdot 10 \cdot \ldots \cdot (2 + 4(n-1))}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (5 + 3(n-1))} = \frac{2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n-2)}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}.$$

We will use the Ratio test to determine the convergence or divergence of this infinite series.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n-2) \cdot (4n-2)}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2) \cdot (3n+5)}}{\dfrac{2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n-2)}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}} = \frac{(4n-2)}{(3n+5)}.$$

Taking the limit of the absolute value of this ratio as $n \to \infty$ gives a limit of 4/3. As this limit is greater than one, the Ratio test gives that the infinite series is divergent.