Solutions to Homework #6

Problems from Pages 404-407 (Section 7.6)

6. We will use the technique of Separation of Variables to solve the differential equation:

$$\frac{dy}{d\theta} = \frac{e^y \cdot \sin^2(\theta)}{y \cdot \sec(\theta)}.$$

First we separate the independent from the dependent variable.

$$\frac{y}{e^y} \cdot dy = \frac{\sin^2(\theta)}{\sec(\theta)} \cdot d\theta.$$

We can rearrange this to make the integrals easier to compute.

$$\int y \cdot e^{-y} \cdot dy = \int \sin^2(\theta) \cdot \cos(\theta) \cdot d\theta$$

We can use integration by parts to integrate the left hand side of this equation and a u-substitution to integrate the right hand side of this equation. Computing these integrals gives:

$$-y \cdot e^{-y} - e^{-y} = \tfrac{1}{3} \cdot \sin^3(\theta) + C.$$

It is not possible to rearrange this equation to make *y* the subject, but this equation is enough to implicitly define *y* as a function of *θ*.

8. We will use the technique of Separation of Variables to solve the differential equation:

$$\frac{dz}{dt} = -e^{t+z} = -e^t \cdot e^z.$$

First we separate the independent from the dependent variable.

$$\frac{1}{e^z} \cdot dz = -e^t \cdot dt.$$

Integrating both sides of this equation gives:

$$\int \frac{1}{e^z} \cdot dz = -\int e^t \cdot dt$$

$$-e^{-z} = -e^{t} + C.$$

Now we can solve to make z the subject of this equation.

$$z = -\ln\left(e^{t} - C\right).$$

12. We will use the technique of Separation of Variables to solve the differential equation:

$$\frac{dP}{dt} = \sqrt{P \cdot t} = \sqrt{P} \cdot \sqrt{t}.$$

First we separate the independent from the dependent variable.

$$\frac{1}{\sqrt{P}} \cdot dP = \sqrt{t} \cdot dt.$$

Integrating both sides of this equation gives:

$$\int \frac{1}{\sqrt{P}} \cdot dP = \int \sqrt{t} \cdot dt$$

$$2 \cdot \sqrt{P} = \tfrac{2}{3} \cdot t^{\frac{3}{2}} + C.$$

To evaluate the constant C we can use the supplied function value $P(1) = 2$. Doing this:

$$2 \cdot \sqrt{2} = \tfrac{2}{3} \cdot (1)^{\frac{3}{2}} + C$$

$$C = 2 \cdot \sqrt{2} - \tfrac{2}{3} \cdot (1)^{\frac{3}{2}}.$$

Now we can plug in the value of C and solve to make P the subject of this equation.

$$P = \left(\frac{\tfrac{2}{3} \cdot t^{\frac{3}{2}} + 2 \cdot \sqrt{2} - \tfrac{2}{3}}{2}\right)^{2}.$$

36. **(a)** The function we are interested in here is $x(t)$, which is defined to be the amount of new currency in circulation at time t. The problem does not specify the units that $x(t)$ is expressed in so we will make the assumption that $x(t)$ is expressed in units of billions of dollars.

In the problem we are told that the initial value is $x(0) = 0$ so devising that part of the model is not a problem.

Initial value: $x(0) = 0$.

The template for the differential equation will be:

$$\frac{dx}{dt} = (\text{Rate in}) - (\text{Rate out}).$$

To determine the "Rate In" we have to determine how much old currency is received by banks and replaced by new currency. The total amount of currency in circulation is \$10 billion and of this, $x(t)$ is new currency. This means that the amount of old currency still in circulation will be $10 - x(t)$ and the fraction of circulating currency that is old will be:

$$\frac{10 - x(t)}{10}.$$

Now, of the \$50 million (= \$0.05 billion) received by banks each day, the amount of it that is old currency is given by:

$$\left(\frac{10 - x(t)}{10}\right) \cdot (0.05) = 0.05 - 0.005 \cdot x(t) = -0.005 \cdot (x(t) - 10).$$

This figure gives the rate at which banks replace old currency with new, so it is the rate at which new currency enters circulation.

The rate at which new currency leaves circulation is zero as new currency is not being removed from circulation.

Putting all of this together gives the differential equation that represents the situation.

Differential equation: $\dfrac{dx}{dt} = -0.005 \cdot (x(t) - 10).$

(b) We will solve the initial value problem using the technique of Separation of Variables. Doing this:

$$\int \frac{1}{x - 10} \cdot dx = \int -0.005 \cdot dt$$

$$\ln(|x - 10|) = -0.005 \cdot t + C$$

$$x - 10 = A \cdot e^{-0.005 \cdot t} \qquad \text{where } A = \pm e^{C}.$$

$$x = 10 + A \cdot e^{-0.005 \cdot t}.$$

To determine the value of the constant A we can plug in the initial condition $x(0) = 0$. Doing this gives $A = -10$ and the final formula for $x(t)$:

$$x = x(t) = 10 - 10 \cdot e^{-0.005 \cdot t}.$$

(c) To determine when 90% of the bills in circulation will be new currency we can set $x(t) = 9$ and solve for t. Doing this:

$$9 = 10 - 10 \cdot e^{-0.005 \cdot t}$$

$$-1 = -10 \cdot e^{-0.005 \cdot t}$$

$$\tfrac{1}{10} = e^{-0.005 \cdot t}$$

$$t = \frac{\ln\left(\frac{1}{10}\right)}{-0.005} = 460.52 \text{ days.}$$

38. The differential equation that we have to work with in this problem is:

$$\frac{dy}{dt} = -\left(8.875 \times 10^{-9}\right) \cdot y \cdot \left(y - 8 \times 10^{7}\right).$$

To solve this equation we can use the techniques of Separation of Variables and partial fractions.

$$\int \frac{1}{y \cdot \left(y - 8 \times 10^{7}\right)} \cdot dy = \int -\left(8.875 \times 10^{-9}\right) \cdot dt$$

To integrate the left hand side of this equation we must break the rational function into two partial fractions. Doing this gives:

$$\frac{1}{y \cdot \left(y - 8 \times 10^{7}\right)} = \frac{A}{y} + \frac{B}{y - 8 \times 10^{7}} = \frac{\frac{-1}{8 \times 10^{7}}}{y} + \frac{\frac{1}{8 \times 10^{7}}}{y - 8 \times 10^{7}},$$

so that:

$$\int \frac{1}{y \cdot \left(y - 8 \times 10^{7}\right)} \cdot dy = \tfrac{-1}{8 \times 10^{7}} \cdot \ln\left(|y|\right) + \tfrac{1}{8 \times 10^{7}} \cdot \ln\left(\left|y - 8 \times 10^{7}\right|\right) = \tfrac{1}{8 \times 10^{7}} \cdot \ln\left(\left|\frac{y - 8 \times 10^{7}}{y}\right|\right).$$

Putting this together with the integral of the right hand side of the original integral equation gives:

$$\tfrac{1}{8\times 10^7}\cdot\ln\left(\left|\frac{y-8\times 10^7}{y}\right|\right)=-\left(8.875\times 10^{-9}\right)\cdot t+C.$$

Rearranging this equation to make y the subject and using $A=\pm e^C$ gives:

$$y=\frac{8\times 10^7}{1-A\cdot e^{-0.71\cdot t}}.$$

(a) We are given $y(0) = 2\times 10^7$. Using this in the above formula for y allows us to solve for A. Doing this gives:

$$2\times 10^7=\frac{8\times 10^7}{1-A\cdot e^{(-0.71)\cdot(0)}}$$

$$1-A=\frac{8}{2}$$

$$A=-3.$$

So, the formula for y is: $$y=\frac{8\times 10^7}{1+3\cdot e^{-0.71\cdot t}}.$$

(c) To determine when the biomass reaches 4×10^7 we can plug this into the formula for y and solve for t. Doing this gives:

$$4\times 10^7=\frac{8\times 10^7}{1+3\cdot e^{-0.71\cdot t}}$$

$$1+3\cdot e^{-0.71\cdot t}=2$$

$$e^{-0.71\cdot t}=\tfrac{1}{3}$$

$$t=\frac{\ln\left(\frac{1}{3}\right)}{-0.71}\approx 1.55 \text{ years.}$$

40. The differential equation and function values in this problem are as follows:

$$\frac{dP}{dt}=k\cdot P\cdot\left(P-10000\right) \qquad P(0)=400 \qquad P(1)=1200,$$

where $P(t)$ is the number fish in the lake after t years.

(a) The steps involved in solving the differential equation are very similar to those employed in the previous problem. The techniques used are Separation of Variables and partial fractions. The steps involved in doing this are shown below.

$$\int \frac{1}{P\cdot(P-10000)}\cdot dP = \int k\cdot dt$$

$$\tfrac{-1}{10000}\cdot\ln(|P|)+\tfrac{1}{10000}\cdot\ln(|P-10000|) = k\cdot t + C$$

$$\ln\left(\left|\frac{P-10000}{P}\right|\right) = k\cdot t + C$$

$$\frac{P-10000}{P} = Ae^{10000\cdot k\cdot t} \text{ where } A = \pm e^{C}.$$

$$P-10000 = P\cdot Ae^{10000\cdot k\cdot t}$$

$$P - P\cdot Ae^{10000\cdot k\cdot t} = 10000$$

$$P\cdot\left[1-Ae^{10000\cdot k\cdot t}\right] = 10000$$

$$P = \frac{10000}{1-Ae^{10000\cdot k\cdot t}}.$$

To determine the value of the constant A we can use the initial value $P(0) = 400$.

$$400 = \frac{10000}{1-Ae^{(10000\cdot k)\cdot(0)}} = \frac{10000}{1-A}$$

so that $A = -24$. To determine the value of the constant k we can use the additional function value $P(1) = 1200$. Doing this gives:

$$1200 = \frac{10000}{1+24e^{(10000\cdot k)\cdot(1)}}$$

$$e^{(10000\cdot k)\cdot(1)} = \frac{\frac{10000}{1200}-1}{24}.$$

Taking logarithms and making k the subject of the equation gives:

$$k = \frac{1}{10000} \cdot \ln\left(\frac{\frac{10000}{1200} - 1}{24}\right) = -0.0001185624,$$

and the final formula for $P(t)$ is:

$$P = P(t) = \frac{10000}{1 + 24e^{-1.185624 \cdot t}}.$$

(b) To calculate when the number of fish reaches 5000, we can plug 5000 for P in the above formula and solve for t. Doing this gives:

$$5000 = \frac{10000}{1 + 24e^{-1.185624 \cdot t}}$$

$$e^{-1.185624 \cdot t} = \frac{\frac{10000}{5000} - 1}{24}$$

$$t = \frac{-1}{1.185624} \ln\left(\left|\frac{\frac{10000}{5000} - 1}{24}\right|\right) \approx 2.68 \text{ years.}$$

44. To solve this problem we will set up a differential equation and an initial value to define $y(t)$, the amount of carbon dioxide in the room after t minutes. The units of $y(t)$ will be cubic meters.

Initial Value: $y(0) = 0.27 \text{ m}^3$.

We can find the initial value by noting that the room has a volume of 180 m^3 and that 0.15% of the air in the room is carbon dioxide. Dividing 0.15 by 100 and multiplying by 180 gives the result of 0.27 m^3.

The template for the differential equation will be:

$$\frac{dy}{dt} = (\text{Rate in}) - (\text{Rate out}).$$

To determine the "Rate In" we have to determine the rate at which carbon dioxide enters the room. To get this we can multiply the flow rate into the room (2 m^3 per minute) by the fraction of this air that is carbon dioxide (0.05 divided by 100, or 0.0005).

To determine the "Rate Out" we have to determine the rate at which carbon dioxide leaves the room. To get this we can multiply the flow rate out of the room (2 m^3 per minute) by the fraction of air in the room that is carbon dioxide. This is $\frac{y(t)}{180}$.

Putting all of this together gives the following differential equation.

Differential Equation: $\frac{dy}{dt} = 0.001 - \frac{2y}{180} = \frac{-1}{90} \cdot (y - 0.09)$.

We will use the technique of Separation of Variables to find a formula for $y(t)$.

$$\int \frac{1}{y - 0.09} \cdot dy = \int \tfrac{-1}{90} \cdot dt$$

$$\ln(|y - 0.09|) = \tfrac{-1}{90} \cdot t + C$$

$$y = 0.09 - Ae^{\frac{-1}{90} \cdot t} \qquad \text{where } A = \pm e^C.$$

We can use the initial value $y(0) = 0.27$ m^3 to determine the value of the constant A. Doing this gives $A = -0.18$ so that the final formula for $y(t)$ is:

$$y = 0.09 + 0.18e^{\frac{-1}{90} \cdot t}.$$

To find a formula for the percentage, $P(t)$, of the air in the room that is carbon dioxide, we can divide $y(t)$ by 180 and then multiply by 100. This gives:

Percentage: $P(t) = \frac{0.09 + 0.18e^{\frac{-1}{90} \cdot t}}{180} \cdot 100 = 0.05 + 0.1e^{\frac{-1}{90} \cdot t}$

In the long run (i.e. as $t \to \infty$) the exponential term in $P(t)$ will get closer and closer to zero. This means that in the long run, the percentage of air in the room that is carbon dioxide will approach 0.05%.

46. To answer the questions in this problem we need to find a formula for $A(t)$, the amount of salt in the tank (measured in kilograms) after t minutes. To do this we need to identify an initial value, $A(0)$ and a differential equation.

Initial Value: $A(0) = 0$.

The tank contains no salt at the beginning (it contains only pure water).

Differential Equation:

$$\frac{dA}{dt} = (0.05)(5) + (0.04)(10) - \frac{A(t)}{1000}(15) = 0.65 - 0.015 \cdot A(t).$$

The rate at which salt flows into the tank is given by multiplying the first stream's flow rate of 5 liters per minute by the salt content of 0.05 kilograms of salt per liter and adding this to the product of the second stream's flow rate of 10 liters per minute and its salt content of 0.04 kilograms per liter. The rate at which the salt flows out of the tank is the product of the concentration of salt in the tank and the outward flow rate of 15 liters per minute.

To find a formula for $A(t)$, we can solve the differential equation using the technique of Separation of Variables. Doing this gives:

$$\frac{dA}{dt} = -0.015 \cdot (A(t) - 43.33)$$

$$\int \frac{1}{A - 43.33} \cdot dA = \int -0.015 \cdot dt$$

$$\ln(|A - 43.33|) = -0.015 \cdot t + C$$

$$A - 43.33 = B \cdot e^{-0.015 \cdot t} \qquad \text{where } B = \pm e^{C}.$$

Plugging in $A = 0$ and $t = 0$ gives $B = 43.33$ and the formula for $A(t)$:

$$A = 43.33 - 43.33 \cdot e^{-0.015 \cdot t}.$$

To answer the two questions posed in the textbook, we can plug specific values of t into this formula.

(a) $A(t) = 43.33 - 43.33 \cdot e^{-0.015 \cdot t}$ kg.

(b) $A(60) = 43.33 - 43.33 \cdot e^{(-0.015) \cdot (60)} = 25.72$ kg.