Solutions to Homework #5

Problems from Pages 383-384 (Section 7.4)

6. The curve in this problem is defined by the equation:

$$y=\frac{x^2}{2}-\frac{\ln(x)}{4}$$

and we are interested in the part of the curve between $x = 2$ and $x = 4$. The derivative is given by:

$$\frac{dy}{dx}=x-\frac{1}{4x},$$

and the integral for the arc length will be:

$$\text{Arc length} = \int_2^4 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx=\left[\frac{4x\left(2x^2+\ln(x)\right)\sqrt{\frac{8x^2+16x^4+1}{x^2}}}{16\left(1+4x^2\right)}\right]_2^4=6+\frac{\ln(2)}{4}.$$

18. The curve in this problem is the ellipse defined by the equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Rearranging to make y the subject of this equation gives: $y=\pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$. The derivative of y with respect to x is given by:

$$y=\pm\frac{b^2x}{a^2}\cdot\left(b^2\left(1-\frac{x^2}{a^2}\right)\right)^{\frac{-1}{2}}=\pm\frac{bx}{a\sqrt{a^2-x^2}}.$$

The integral for the length of the portion of the ellipse that lies above the x-axis is:

$$\int_{-a}^{a}\sqrt{1+\left(\frac{bx}{a\sqrt{a^2-x^2}}\right)^2}\,dx=\int_{-a}^{a}\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^4-a^2x^2}}\,dx.$$

The total arc length of the entire ellipse (including the half above the x-axis and the half below the x-axis) will be twice this integral.

20. There are many different ways to attack this problem. The solution given here has the virtue of being relatively short but may not relate well in your mind to what we studied when we learned about arc length in lecture. If so, come to office hours to discuss this (or alternative) methods of solution.

Here we will interchange the roles of x and y in the usual arc length integration formula so that:

$$\text{Arc length} = \int_{y=a}^{y=b} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

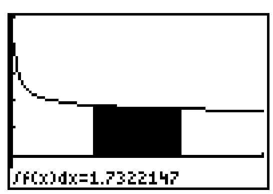
The integral that we will approximate using Simpson's rule will then be:

$$\text{Arc length} = \int_{y=1}^{y=2} \sqrt{1+\left(1+\frac{1}{2\sqrt{y}}\right)^2}\, dy.$$

The approximate value of this integral given by Simpson's rule with $n = 10$ is:

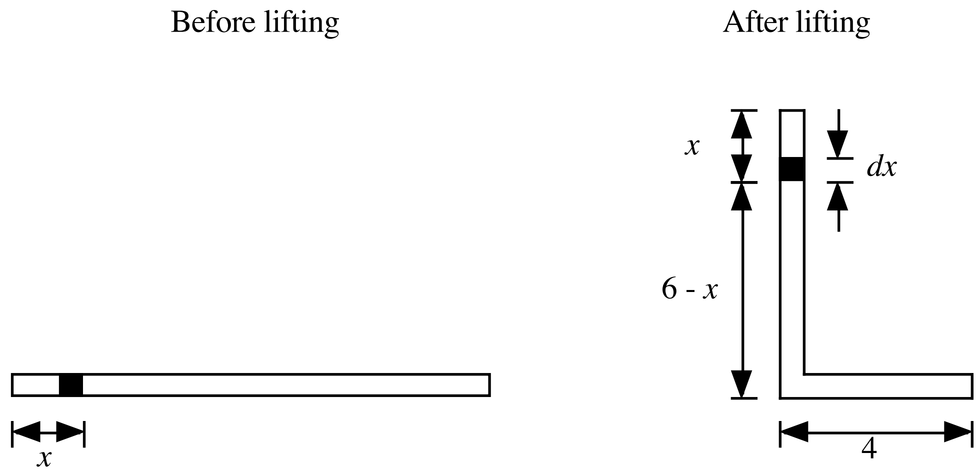
$$\text{Arc length} \approx 1.732215.$$

(This figure was obtained by calculating a midpoint and a trapezoid sum for the above integral using $n = 5$ and then combining one third of the trapezoid total with two thirds of the midpoint total.) Calculator integration on a TI-83 Plus (see below) gave the result shown below, which is almost exactly the same value furnished by Simpson's rule.



Problems from Pages 394-397 (Section 7.5)

10. In writing this solution I assume that at the end of the chain lifting operation, there are still four meters of chain lying on the ground in a horizontal position and the top of the suspended part of the chain is six meters off the ground (see diagrams below). If your assumptions about the final configuration of the chain are different then you may have set up the problem in a different way.



Let's begin by focusing on the work done to raise the small piece of chain (shaded in the above diagram) that lies between x and $x + dx$. The vertical distance that this piece of chain moves is the distance $6 - x$ meters. The force exerted by gravity on this small piece of chain is g times the mass of the small piece of chain. This force is given by $(9.8)\left(\frac{80}{10}\right)dx$. Multiplying these two together gives the work to raise the small piece of chain:

$$\text{Work to raise small piece of chain} = (6 - x)(9.8)\left(\tfrac{80}{10}\right)dx.$$

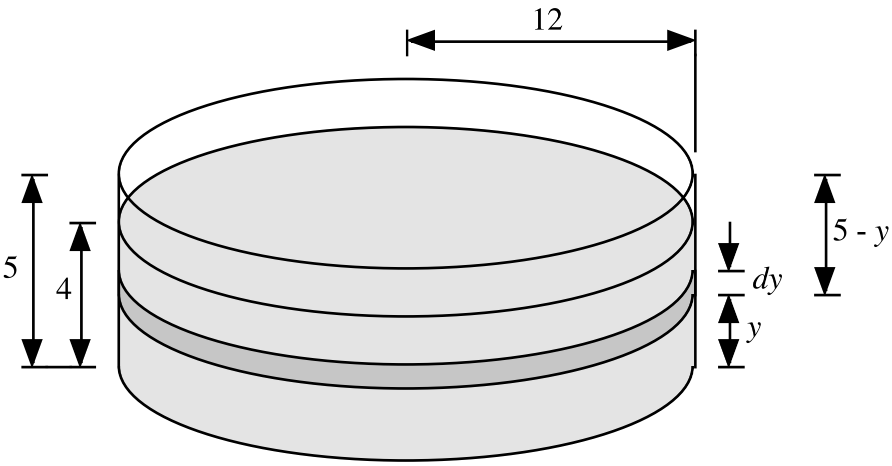
The work done to raise the first six meters of chain can be expressed as the Riemann sum:

$$\text{Total work} = \text{Lim}_{N\to\infty}\sum_{k=0}^{N-1}\left(6 - k\cdot\tfrac{6-0}{N}\right)\cdot(9.8)\cdot\left(\tfrac{80}{10}\right)\cdot k\cdot\tfrac{6-0}{N}.$$

As we take the limit as $N \to \infty$, the total work is given by the definite integral:

$$\text{Total work} = \int_0^6 (6 - x)(9.8)\left(\tfrac{80}{10}\right)dx = 78.4\left[6x - \tfrac{1}{2}x^2\right]_0^6 = 1411.2 \text{ Nm}.$$

16. The swimming pool is illustrated hereafter. To calculate the amount of work needed to empty the pool, we will slice the water up into pieces. The defining characteristic of these pieces is that each molecule of water contained in the piece move approximately the same vertical distance to leave the pool. This means that the pieces will be thin, horizontal slices of water as shown in the diagram (dark gray).



The distance moved by all the water molecules in the dark gray horizontal slice will be $5 - y$ feet.

To get the force exerted by gravity on the dark gray slice we need to multiply the density of 62.5 pounds per cubic foot by the volume of the dark gray slice (expressed in cubic feet). Note that we do not have to multiply by 9.8 m/s^2 (or its imperial equivalent of 32 ft/s^2) because pounds are already units of force.

The volume of the dark gray slice (a disk with radius 12 feet and thickness dy feet) is $\pi \cdot 12^2 \cdot dy$ cubic feet so the force exerted by gravity is $62.5 \cdot \pi \cdot 12^2 \cdot dy$ pounds.

Multiplying distance moved by force gives work, so the work (in units of foot pounds) required to remove the dark gray slice of water from the swimming pool will be:

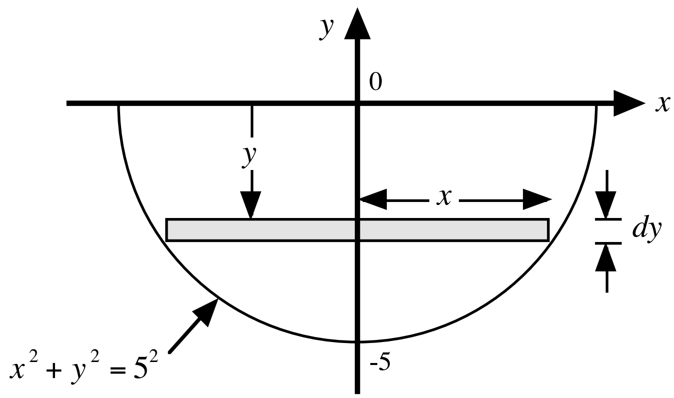
$$\text{Work to remove dark gray slice} = (5 - y) \cdot 62.5 \cdot \pi \cdot 12^2 \cdot dy.$$

The total amount of work to remove all of the water from the pool will be the integral of this from the value of y where the water begins ($y = 0$) to the value of y where the water ends ($y = 4$). This integral is:

$$\text{Total work} = \int_0^4 (5 - y) \cdot 62.5 \cdot \pi \cdot 12^2 \cdot dy = 9000 \cdot \pi \cdot \left[5y - \tfrac{1}{2}y^2\right]_0^4 = 3392292.0066 \text{ ft-lbs.}$$

18. The method of slicing the water in the hemispherical tank is exactly the same as the cylindrical swimming pool featured in the previous problem. The main difference between this problem and the previous one is that the radius of this tank varies with y (rather than staying constant).

A sideways view of the hemispherical tank (with a horizontal slice of water shown in gray) is given in the following diagram. From the appearance of the diagram, the vertical distance moved by the slice as it is pumped out of the tank is $-y$.



As we are again dealing with imperial units, the force exerted on the slice by gravity will be given by 62.5 pounds per cubic foot times the volume of the slice. The slice is shaped like a disk with radius x and thickness dy so the volume of the slice will be:

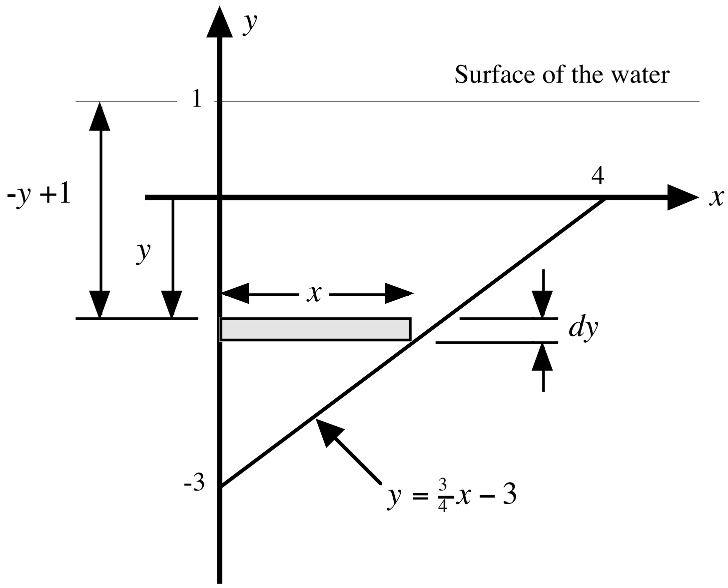
$$\text{Volume of the slice} = \pi \cdot x^2 \cdot dy = \pi \cdot \left(5^2 - y^2\right) \cdot dy.$$

The force exerted by gravity on the slice will be $62.5 \cdot \pi \cdot \left(5^2 - y^2\right) \cdot dy$ and the work done (in units of foot pounds) to remove this slice from the tank will be $-y \cdot 62.5 \cdot \pi \cdot \left(5^2 - y^2\right) \cdot dy$.

The total work to remove all of the water from the tank will be the integral of this from the y-value where the water begins ($y = -5$) to the y-value where the water ends ($y = 0$). This integral is as follows:

$$\text{Total work} = \int_{-5}^{0} -y \cdot 62.5 \cdot \pi \cdot \left(5^2 - y^2\right) \cdot dy = 196.349 \cdot \left[\tfrac{-25}{2} y^2 + \tfrac{1}{4} y^4\right]_{-5}^{0} = 30679.62 \text{ ft-lbs.}$$

24. To compute the hydrostatic force exerted on one side of the triangular metal plate, we need to slice the triangular plate into pieces so that each point of each piece experiences approximately constant pressure. As pressure depends on depth, these pieces will be horizontal rectangles as shown in the following diagram.



The force exerted on the horizontal rectangle is the product of pressure and area. The pressure is the product of the density of water (62.5 pounds per cubic foot) and the depth below the surface. As shown in the previous diagram, the depth below the surface is given by $1 - y$, so the pressure is $62.5 \cdot (1 - y)$.

The area of the rectangle is the width of the rectangle multiplied by dy. The width of the rectangle can be written as x so that the area of the rectangle is $x \cdot dy$. However, this will not be the most convenient expression for the area when we set up the integral to give the total force. To create a more convenient expression, we need to substitute for the x in this area formula.

The tool that enables us to make this substitution is an equation for the straight line that forms the diagonal side of the rectangle in the previous diagram. This is the equation for the line that joints the points $(0, -3)$ and $(4, 0)$ which is:

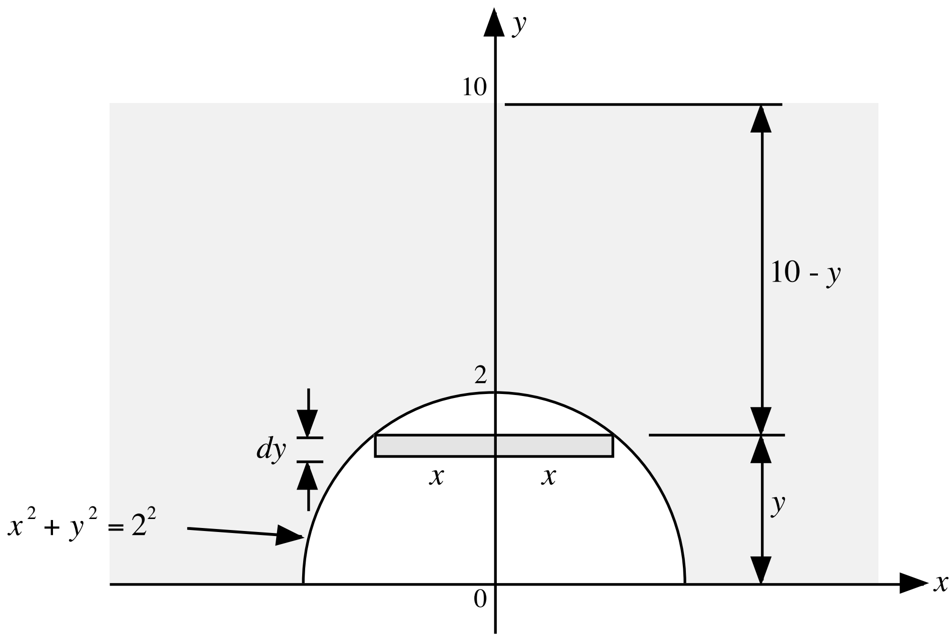
$$y = \tfrac{3}{4}x - 3 \qquad \text{or} \qquad x = \tfrac{4}{3}(y + 3).$$

With this expression, the area of the horizontal rectangle is $\tfrac{4}{3}(y+3) \cdot dy$ and the force exerted on the horizontal rectangle is $62.5 \cdot (1 - y) \cdot \tfrac{4}{3} \cdot (y + 3) \cdot dy$.

The total force exerted on one side of the triangular plate will be the integral of this expression from the y-value where the plate begins ($y = -3$) to the y-value where the plate ends ($y = 0$). This integral is:

$$\int_{-3}^{0} 62.5 \cdot (1 - y) \cdot \tfrac{4}{3} \cdot (y + 3) \cdot dy = 750 \text{ lbs.}$$

30. As with the previous problem, we will slice the area of the gate into horizontal rectangles as each point on a thin horizontal rectangle experiences almost the same pressure. This is because the points are all at about the same depth and pressure depends on the depth of the water. A diagram showing the situation with a thin, horizontal rectangle drawn on the circular gate is shown below.



As in the previous example, we will calculate the force exerted on a single horizontal rectangle and then integrate this to find the total hydrostatic force.

The pressure exerted on the rectangle shown in the previous diagram is the product of gravitational acceleration, the density of water and the depth. In symbols this is $9.8 \cdot 1000 \cdot (10 - y)$ N/m^2.

The area of the horizontal rectangle is given by $2x \cdot dy = 2 \cdot \sqrt{4 - y^2} \cdot dy$. Multiplying pressure and area gives the following expression for the force exerted on the horizontal rectangle:

Force exerted on horizontal rectangle = $9.8 \cdot 1000 \cdot (10 - y) \cdot 2 \cdot \sqrt{4 - y^2} \cdot dy$.

The total force exerted on the semi-circular gate is the integral of this expression from the y-value where the gate begins ($y = 0$) and the y-value where the gate ends ($y = 2$). This integral is evaluated using trigonometric substitution ($y = 2 \cdot \sin(\theta)$) and a u-substitution ($u = 4 - y^2$) to obtain:

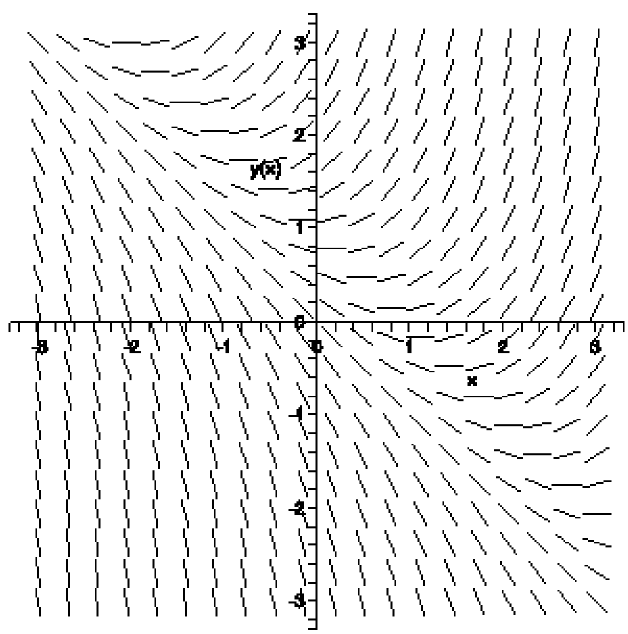
$$\text{Total force} = \int_0^2 9.8 \cdot 1000 \cdot (10 - y) \cdot 2 \cdot \sqrt{4 - y^2} \cdot dy = 9800 \cdot \left(20\pi - \tfrac{16}{3}\right) \text{ N.}$$

Problems from Pages 404-407 (Section 7.6)

22. If you draw the slope field corresponding to the differential equation:

$$\frac{dy}{dx} = x + y - 1,$$

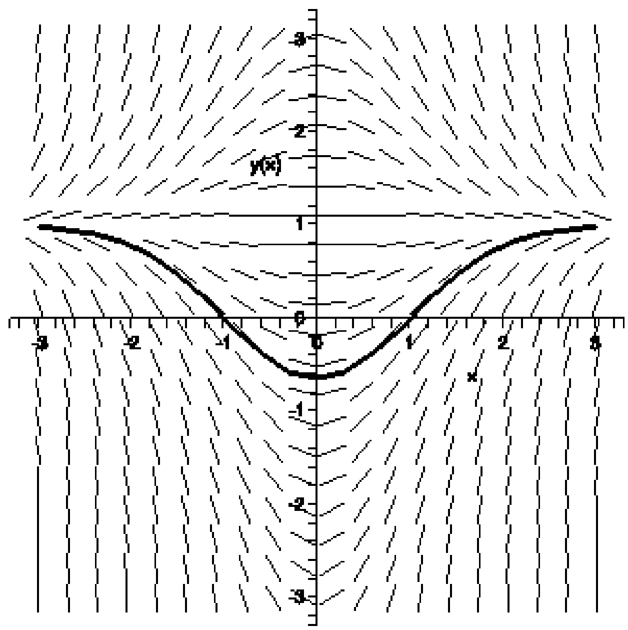
you will obtain the picture shown below. From the alternatives shown at the bottom of page 405, Diagram IV is the closest match.



32. The slope field corresponding to the differential equation:

$$\frac{dy}{dx} = x - x \cdot y,$$

is shown in the diagram below. The curve shown on the slope field is the solution curve that passes through the point (1, 0).



y(x)
x