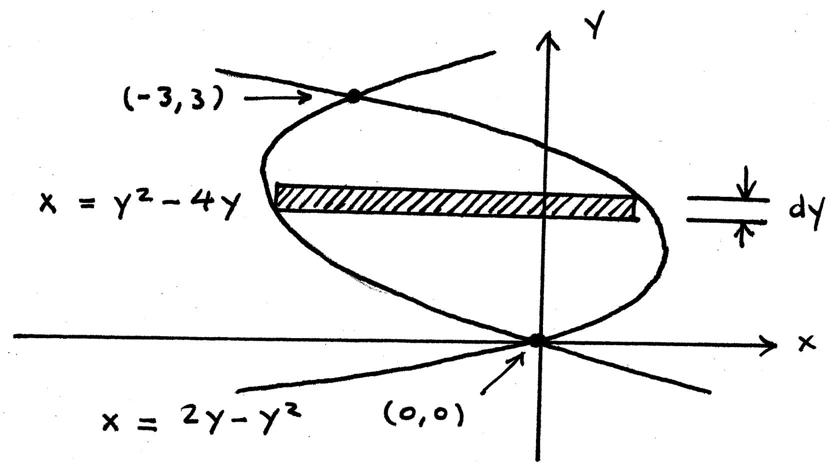
SOLUTIONS FOR HW #4.

Problems from p. 361 – 362 :

4. To find the area between $x = y^2 - 4y$ and $x = 2y - y^2$, we can divide the area into horizontal rectangles.



The area of the shaded rectangle is :

$$\text{Area} = \left(2y - y^2 - (y^2 - 4y)\right) \cdot dy$$
$$= (6y - 2y^2)\, dy$$

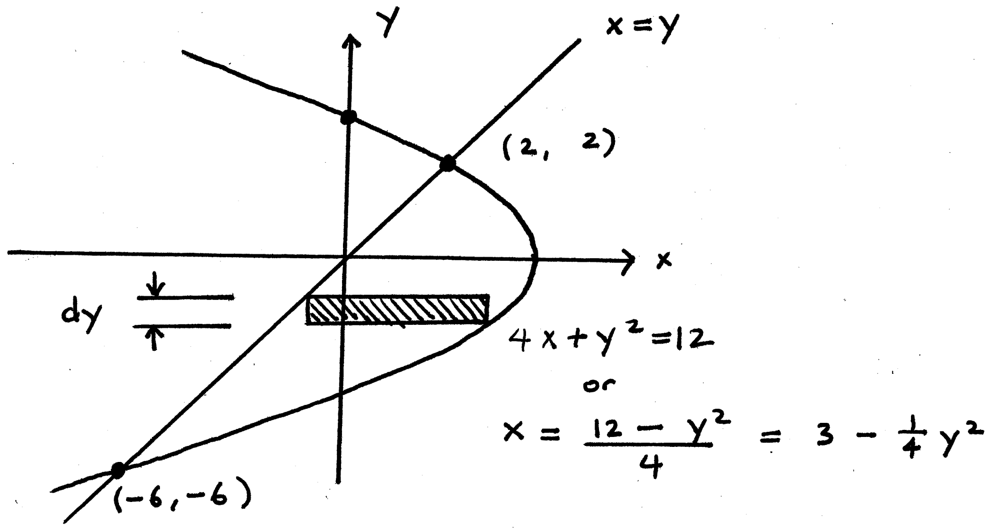
The total area bounded by the two curves is :

$$\text{Total area} = \int_0^3 (6y - 2y^2)\, dy$$
$$= \left[3y^2 - \frac{2}{3}y^3\right]_0^3$$
$$= 9.$$

12. We have to find the area bounded by:

$$4x + y^2 = 12 \qquad x = y.$$

We will start with a sketch of the area to see how to break it up.



The calculation will be easiest (and involve only integral if we break up the area into horizontal rectangles.

$$\text{Area of rectangle} = \left(3 - \tfrac{1}{4}y^2 - y\right) \cdot dy$$

To determine the limits of integration, we need to find the intersection of $4x+y^2 = 12$ and $x=y$. Using $x=y$ to replace $4x+y^2=12$ with:

$$4y + y^2 = 12$$

and solving using the quadratic formula gives:

$$y = \frac{-4 \pm \sqrt{4^2 - 4(1)(12)}}{2} = 2, -6.$$

$$\text{Total area} = \int_{-6}^{2} \left(3 - \tfrac{1}{4}y^2 - y\right) dy$$

$$= \left[3y - \tfrac{1}{12}y^3 - \tfrac{1}{2}y^2\right]_{-6}^{2}$$

$$= \frac{64}{3}$$

26. The version of Simpson's rule most readily applicable to this problem is:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

which is described on page 340 of the textbook.

In this problem, $\Delta x = 20$ cm, so the area of the wing is approximated by:

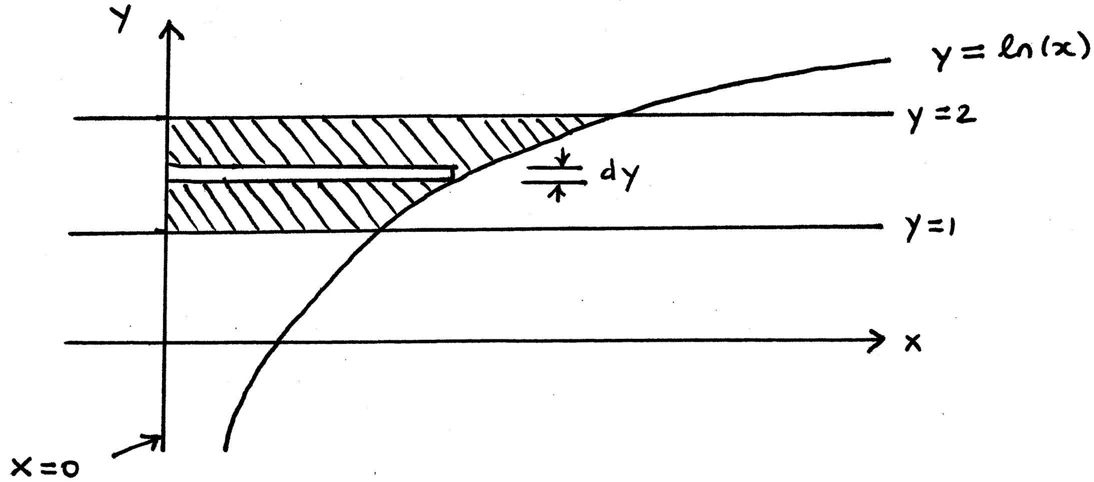
$$\begin{aligned}\text{Area} &\approx \frac{20}{3}\big[5.8 + (4)(20.3) + (2)(26.7) + (4)(29.0) \\ &\quad + (2)(27.6) + (4)(27.3) + (2)(23.8) \\ &\quad + (4)(20.5) + (2)(15.1) + (4)(8.1) + 2.8\big] \\ &\approx 4121 \text{ cm}^2\end{aligned}$$

<u>Problems from p. 370 – p. 373.</u>

4. We need to find the volume generated when the area bounded by:

$$y = \ln(x) \qquad y = 1 \qquad y = 2 \qquad x = 0$$

is rotated around the x-axis. We will start with a sketch of the region to determine how best to slice it up.



The shaded area is revolved around the y-axis. We can compute the volume using a disk approach by dividing the area into horizontal rectangles and then revolving the horizontal rectangles around the y-axis to form disks.

$$\text{Disk volume} = \pi \cdot x^2 \cdot dy$$

To remove the 'x' from the disk volume formula, note that $y = \ln(x)$ so that $x = e^y$ and $x^2 = (e^y)^2 = e^{2y}$. This gives:

$$\text{Disk volume} = \pi \cdot e^{2y} \cdot dy$$

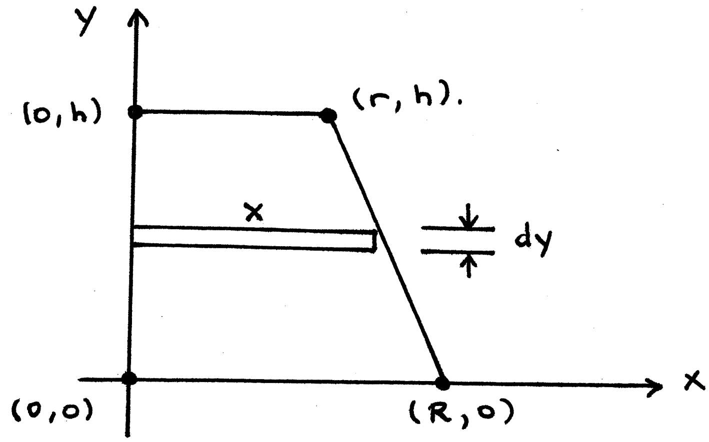
The total volume will then be:

$$\text{Total volume} = \int_1^2 \pi \cdot e^{2y} \cdot dy$$

$$= \left[\frac{\pi}{2} e^{2y} \right]_1^2$$

$$= \frac{\pi}{2} \left(e^4 - e^2 \right)$$

26. To create the dreaded frustrum cone, we can imagine the following area revolved around the y-axis:



The area can be sliced into horizontal rectangles and these rectangles revolved around the y-axis to create disks. The volume of each disk will be:

$$\text{Disk volume} = \pi x^2 \, dy.$$

In order to integrate we need to substitute for the x in the disk volume formula. The tool that will enable us to do this is the equation for the straight line that forms the outer edge of the area, namely the straight line joining (r,h) and $(0,R)$, $y = mx + b$.

$$\text{Slope} = m = \frac{\Delta y}{\Delta x} = \frac{0-h}{R-r} = \frac{-h}{R-r}$$

To find the intercept, plug this slope and $(0,R)$ into $y = mx + b$.

$$0 = \frac{-h}{R-r} \cdot R + b$$

$$b = \frac{hR}{R-r}$$

so, $$y = \frac{-h}{R-r}x + \frac{hR}{R-r}$$

or, $$x = \frac{R-r}{-h}\cdot\left(y - \frac{hR}{R-r}\right)$$

Plugging this into the disk volume formula gives:

$$\text{Disk Volume} = \pi\cdot\left(\frac{R-r}{-h}\right)^2\cdot\left(y - \frac{hR}{R-r}\right)^2\cdot dy.$$

The total volume of the frustrum cone is given by:

$$\text{Total volume} = \int_0^h \pi\left(\frac{R-r}{-h}\right)^2\cdot\left(y - \frac{hR}{R-r}\right)^2 dy$$

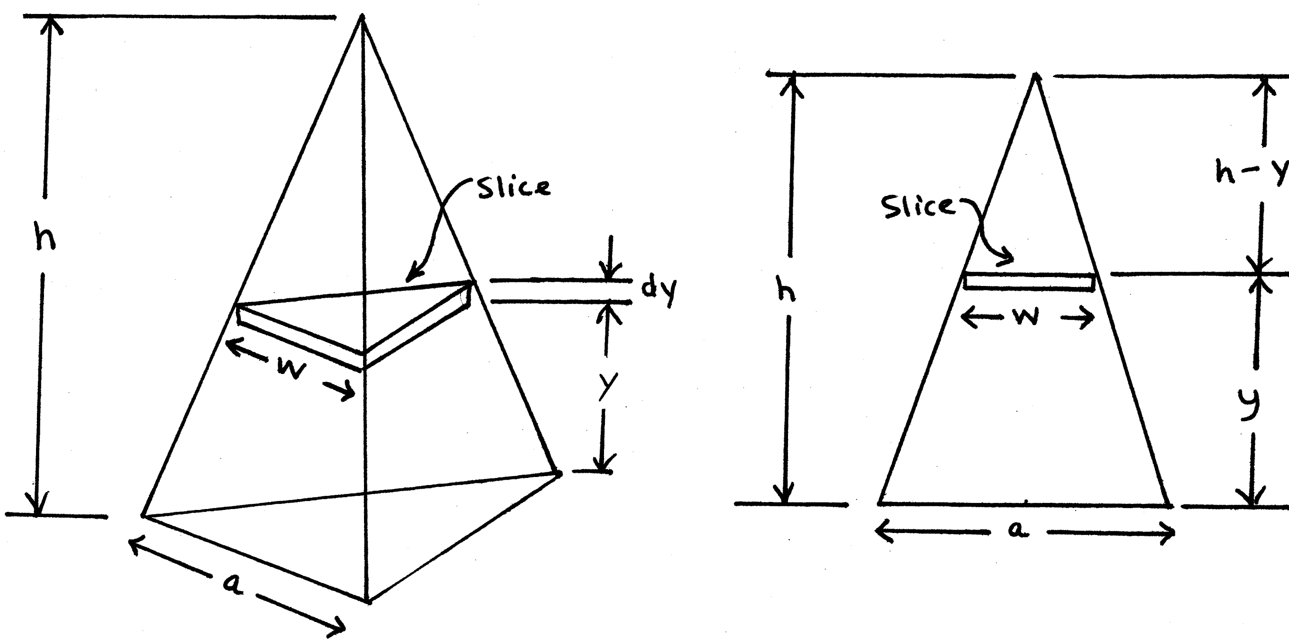
$$= \pi\cdot\frac{(R-r)^2}{h^2}\int_0^h\left(y^2 - \frac{2hR}{R-r}y + \frac{h^2R^2}{(R-r)^2}\right)dy$$

$$= \pi\cdot\frac{(R-r)^2}{h^2}\left[\frac{1}{3}y^3 - \frac{hR}{R-r}y^2 + \frac{h^2R^2}{(R-r)^2}y\right]_0^h$$

$$= \frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$$

30. The area of an equilateral triangle with side length "w" is: $\frac{\sqrt{3}}{4}w^2$.

To see how to set up an integral to compute the volume of the triangular pyramid, imagine slicing the pyramid horizontally.



The slice will be an equilateral triangle with a thickness of dy. The volume of the slice will be:

$$\text{Slice volume} = \frac{\sqrt{3}}{4} w^2 \cdot dy$$

In order to integrate, we must substitute for the w in the slice volume formula. To do this we can use similar triangles:

$$\frac{w}{h-y} = \frac{a}{h}$$

$$w = \frac{a}{h}(h-y)$$

so that:

$$\text{Slice volume} = \frac{\sqrt{3}}{4}\left(\frac{a}{h}\right)^2 (h-y)^2\, dy$$

and:

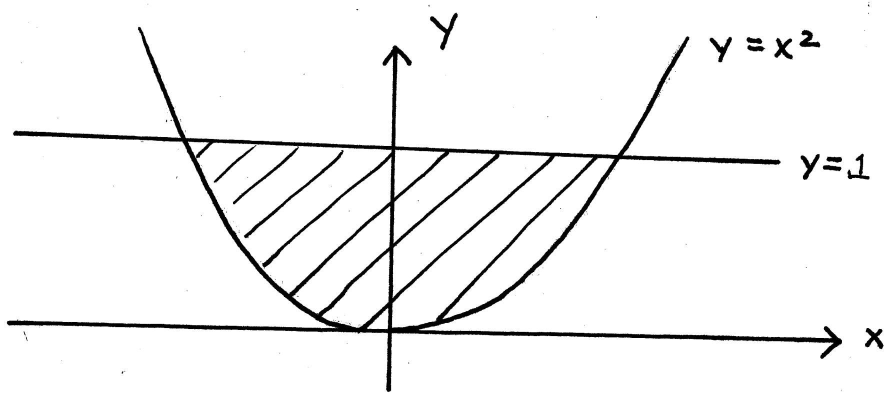
$$\text{Total volume} = \int_0^h \frac{\sqrt{3}}{4}\left(\frac{a}{h}\right)^2 (h-y)^2\, dy$$

$$= \frac{\sqrt{3}}{4}\,\frac{a^2}{h^2}\int_0^h (h^2 - 2hy + y^2)\,dy = \frac{\sqrt{3}}{12}a^2 h.$$

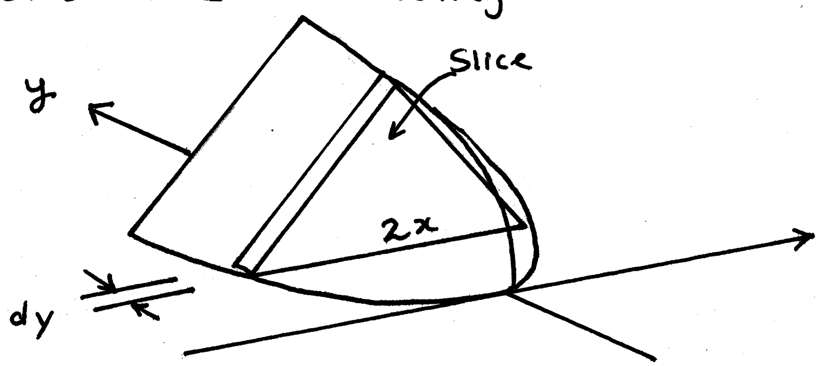
34. We will begin by drawing the region

$$\{(x,y) \mid x^2 \le y \le 1\}$$

to understand what the base of S looks like. This is the region between $y = x^2$ and $y = 1$.



The cross-sections of S, perpendicular to the y-axis, are equilateral triangles, so S resembles the following:



The volume of each slice is given by:

$$\text{Slice volume} = \frac{\sqrt{3}}{4}(2x)^2 \cdot dy$$

$$= \sqrt{3} \cdot x^2 \cdot dy$$

To integrate we must replace the x^2 in the slice volume formula. The relationship $y = x^2$ provides the tool to do this, giving:

$$\text{Slice volume} = \sqrt{3} \cdot y \cdot dy.$$

The total volume is given by:

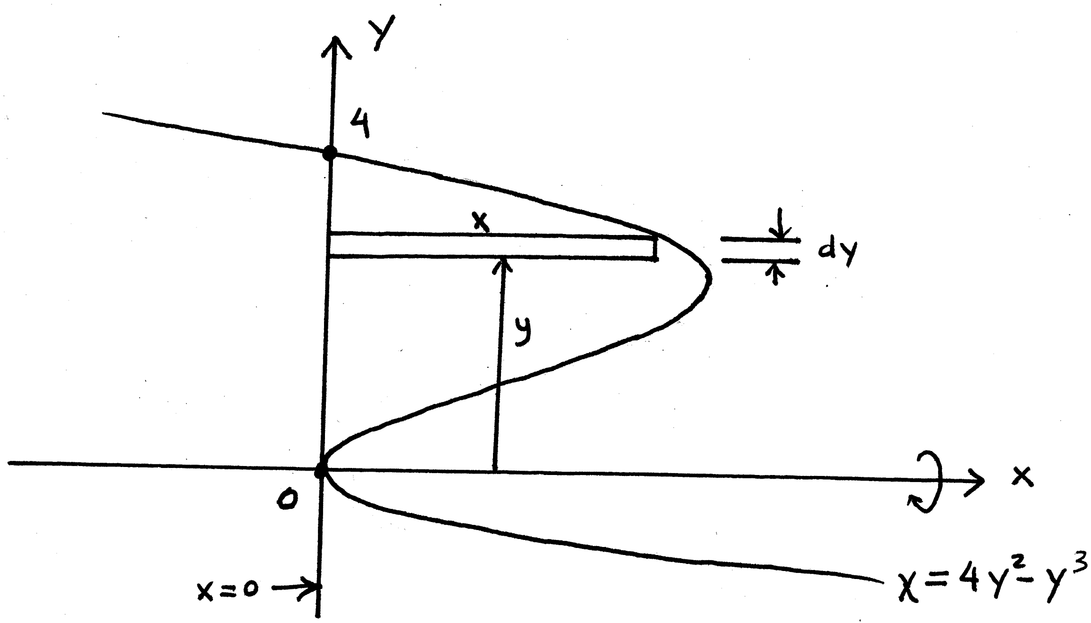
$$\text{Total volume} = \int_0^1 \sqrt{3}\, y \, dy = \frac{\sqrt{3}}{2}.$$

Problems from p. 376 – p. 377.

12. We begin by drawing the region bounded by

$$x = 4y^2 - y^3 \qquad x = 0$$

to understand how we will break it up to compute the integral that will give the volume of revolution. The axis of revolution in this problem is the x-axis.



To calculate the volume using cylindrical shells, it will be easiest to break the area up into horizontal rectangles. When a horizontal rectangle is revolved around the x-axis it will create a shell with volume:

Shell volume $= 2\pi y \cdot x \cdot dy$

To integrate, replace the x in the shell volume formula using $x = 4y^2 - y^3$ and note that the y-values where the area begins and ends are $y = 0$ and $y = 4$. Using this:

$$\text{Total volume} = \int_0^4 2\pi y \cdot (4y^2 - y^3) \cdot dy$$

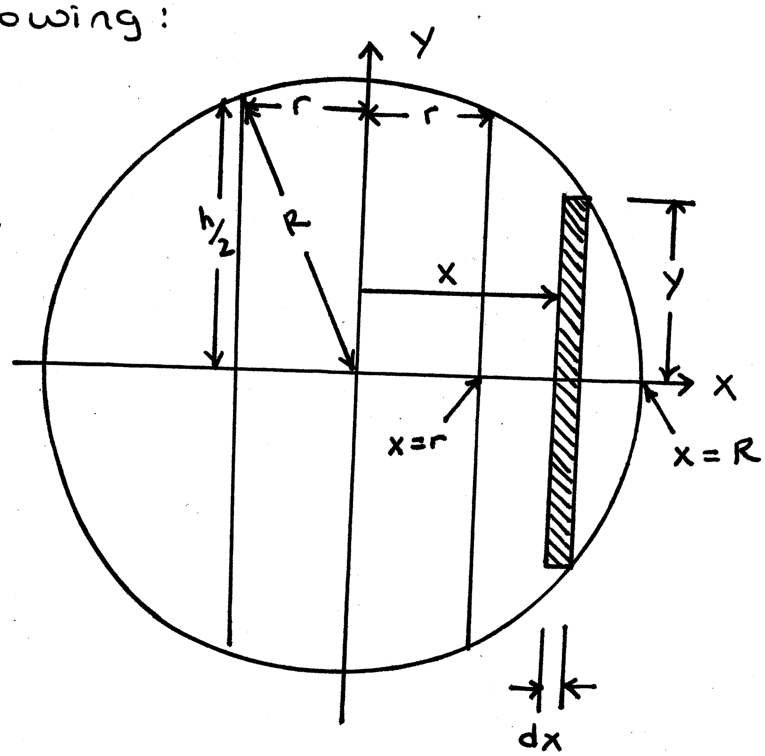
$$= 2\pi \cdot \left[y^4 - \frac{1}{5} y^5 \right]_0^4$$

$$= \frac{512\pi}{5}$$

42 (b). When viewed from the side, the sphere of radius R with the vertical hole of radius r will look something like the following:

$x^2 + y^2 = R^2$

$y = \sqrt{R^2 - x^2}$



By Pythagoras' Theorem:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$R^2 - r^2 = \frac{h^2}{4}$$

To compute the volume using shells, slice the area of the circle that lies between $x = r$ and $x = R$ into vertical rectangles.

Each rectangle is revolved around the y-axis to create a shell with volume:

$$\text{Shell volume} = 2\pi x \cdot (2y) \cdot dx$$
$$= 2\pi x \cdot 2 \cdot \sqrt{R^2 - x^2} \cdot dx$$

using $y = \sqrt{R^2 - x^2}$ to replace the y.

The total volume is given by:

$$\text{Total volume} = \int_r^R 2\pi x \cdot 2 \cdot \sqrt{R^2 - x^2} \cdot dx$$

$$\boxed{\begin{array}{l}\text{u-substitution} \\ u = R^2 - x^2 \\ dx = \dfrac{du}{-2x}\end{array}}$$

$$= \left[-\frac{4\pi}{3} (R^2 - x^2)^{3/2} \right]_r^R$$

$$= \frac{4\pi}{3} (R^2 - r^2)^{3/2}$$

To express the total volume in terms of h, note that $R^2 - r^2 = h^2/4$ so:

$$\text{Total volume} = \frac{4\pi}{3} \left(\frac{h^2}{4} \right)^{3/2} = \frac{\pi h^3}{6}.$$