Solutions to Homework #3

Problems from Pages 343-345 (Section 6.5)

2.(a) The graph of $y = f(x)$ given on page 343 shows that between $x = 0$ and $x = 2$, the function is decreasing and concave up. Under these circumstances,

Right Riemann Sum ≤ Midpoint ≤ Exact integral ≤ Trapezoid ≤ Left Riemann Sum.

This ranking means that the given values will correspond to the following approximation methods:

Left hand Riemann sum = 0.9540

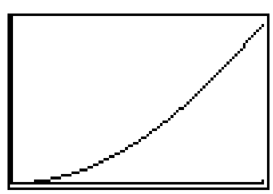
Trapezoid rule = 0.8675

Midpoint rule = 0.8632

Right hand Riemann sum = 0.7811.

2.(b) The exact value of the integral lies between the values given by the Trapezoid and Midpoint rules, namely between 0.8632 and 0.8675.

4.(a) The graph of $f(x) = \sin\left(\frac{x^2}{2}\right)$ for $0 \le x \le 1$ is an increasing, concave up graph as pictured below. Note that your calculator should be in RADIAN mode for this graph to display properly.



This graph is an increasing, concave up graph so the relationship of the values produced by the various approximate integration methods that we are familiar with is as follows.

Left Riemann Sum ≤ Midpoint ≤ Exact integral ≤ Trapezoid ≤ Right Riemann Sum.

So, the left hand Riemann sum and the midpoint rule give underestimates whereas the trapezoid rule and the right hand Riemann sum give overestimates.

4.(b) The ordering is given above.

4.(c) The values of the various estimates of the integral (obtained by executing sum and seq commands on a graphing calculator) are given in the table below.

Approximation method	Result using 5 rectangles
Left hand Riemann sum	0.1187
Midpoint rule	0.1622
Trapezoid rule	0.1666
Right hand Riemann sum	0.2146

The midpoint and trapezoid rules will be closest to the actual value of the integral. The solution manual hazards the guess that the midpoint rule gives a more accurate estimate of the integral in this case.

6. The results of estimating $\int_0^1 e^{-\sqrt{x}}\,dx$ using 6 rectangles are shown in the table given below.

Approximation method	Result using 6 rectangles
Midpoint	0.525100
Simpson	0.533979

The exact value of the integral can be calculated using *u*-substitution and integration by parts. It comes out to be:

$$\int_0^1 e^{-\sqrt{x}}\,dx = 2 - \frac{4}{e} \approx 0.528482.$$

The error in each case is the exact value of the integral minus the approximate value. The errors for midpoint and Simpson are shown in the table given below.

Approximation method	Result using 6 rectangles	Error
Midpoint	0.525100	0.003382
Simpson	0.533979	–0.005497

10. The results of using the specified methods to approximate $\int_0^3 \frac{1}{1+t^2+t^4}\,dt$ with 6 rectangles are given below. These were obtained by executing sum and seq commands on a graphing calculator.

Approximation method	Result using 6 rectangles
Midpoint	0.895478
Trapezoid	0.895122
Simpson	0.898014

12. The results of using the specified methods to approximate $\int_0^4 \sqrt{1+\sqrt{x}}\,dx$ with 68rectangles are given below. These were obtained by executing sum and seq commands on a graphing calculator.

Approximation method	Result using 8 rectangles
Midpoint	6.084778
Trapezoid	6.042985
Simpson	6.061678

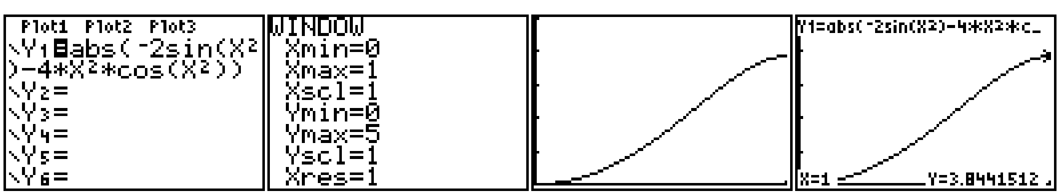
18.(a) The results of using the specified methods to approximate $\int_0^1 \cos\left(x^2\right)dx$ with 8 rectangles are given below. These were obtained by executing sum and seq commands on a graphing calculator.

Approximation method	Result using 8 rectangles
Midpoint	0.902333
Trapezoid	0.905620

18.(b) The second derivative of $f(x)=\cos\left(x^2\right)$ is:

$$f''(x) = -2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right).$$

Graphing the absolute value of the second derivative over the interval [0, 1] gives the results shown below.



The maximum value of the absolute value of the second derivative is less than 3.85, so we will use this as k in the error estimates.

For the Trapezoid rule with $a = 0$, $b = 1$, $N = 8$ and $k = 3.85$ we get:

$$\left|\text{Error}\right| \le \frac{k \cdot (b-a)^3}{12 \cdot N^2} \approx 0.0050130208.$$

For the Midpoint rule with $a = 0$, $b = 1$, $N = 8$ and $k = 3.85$ we get:

$$\left|\text{Error}\right| \le \frac{k \cdot (b-a)^3}{24 \cdot N^2} \approx 0.0025065104.$$

18.(c) For the Trapezoid rule, we solve the following equation for N:

$$\frac{(3.85) \cdot (1-0)^3}{12 \cdot N^2} \le 0.00001,$$

which gives $N \geq 179.118$. So, a minimum of 180 rectangles would be needed when using the Trapezoid rule.

For the Midpoint rule, we solve the following equation for N:

$$\frac{(3.85) \cdot (1-0)^3}{24 \cdot N^2} \le 0.00001,$$

which gives $N \geq 126.655$. So, a minimum of 127 rectangles would be needed when using the Midpoint rule.

26. The distance covered between $t = 0$ and $t = 0.5$ will be given by the integral:

$$\text{Distance} = \int_0^5 v(t)dt.$$

Using Simpson's rule to do this with $\Delta t = 0.5$ gives:

$$\text{Distance} \approx \frac{0.5}{3}\left[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81\right] = 44.735$$

So, the distance covered between t = 0 and t = 0.5 is approximately 44.735 meters.

Problems from Pages 352-354 (Section 6.6)

2.(a) The function $f(x)=\dfrac{1}{2x-1}$ is defined everywhere on the interval [1, 2] so the integral $\displaystyle\int_1^2 \frac{1}{2x-1}dx$ is a "proper" integral.

2.(b) The function $f(x)=\dfrac{1}{2x-1}$ has a vertical asymptote at $x = 0.5$, which is contained in the interval [0, 1]. The integral $\displaystyle\int_0^1 \frac{1}{2x-1}dx$ is an "improper" integral of Type II.

2.(c) The integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2}dx$ has infinite limits of integration. It is an "improper" integral. According to the classification system used by the book, this would be a Type I integral; according to the classification system from class, a Type III.

2.(d) The function $g(x)=\ln(x-1)$ has a vertical asymptote at x =1, which is the endpoint of the interval [1, 2]. The integral $\displaystyle\int_1^2 \ln(x-1)dx$ is an "improper" integral of Type II.

14. To determine the convergence or divergence of $\displaystyle\int_{-\infty}^{\infty} x^2\cdot e^{-x^3}dx$, we must break the integral at a finite x-value (we will use $x = 0$ here) and check the convergence or divergence of each individual piece of the integral.

Convergence or divergence of $\displaystyle\int_0^{\infty} x^2\cdot e^{-x^3}dx$:

$$\int_0^{\infty} x^2\cdot e^{-x^3}dx = \text{Lim}_{a\to\infty}\int_0^{a} x^2\cdot e^{-x^3}dx = \text{Lim}_{a\to\infty}\left[\tfrac{-1}{3}e^{-x^3}\right]_0^a = \tfrac{1}{3}.$$

So $\displaystyle\int_0^{\infty} x^2\cdot e^{-x^3}dx$ converges.

Convergence or divergence of $\displaystyle\int_{-\infty}^{0} x^2\cdot e^{-x^3}dx$:

$$\int_{-\infty}^{0} x^2 \cdot e^{-x^3} dx = \text{Lim}_{a \to -\infty} \int_{a}^{0} x^2 \cdot e^{-x^3} dx = \text{Lim}_{a \to -\infty} \left[\tfrac{-1}{3} e^{-x^3} \right]_a^0 = +\infty.$$

So $\int_{-\infty}^{0} x^2 \cdot e^{-x^3} dx$ diverges and $\int_{-\infty}^{\infty} x^2 \cdot e^{-x^3} dx$ diverges.

42. We will determine the convergence or divergence of $\int_{1}^{\infty} \frac{2 + e^{-x}}{x} dx$ by comparison to a known improper integral. To begin with, note that the "p-integral" $\int_{1}^{\infty} \frac{2}{x} dx$ diverges. Next, observe that:

$$e^{-x} \geq 0,$$

so that,

$$2 + e^{-x} \geq 2,$$

and for $x > 0$,

$$\frac{2 + e^{-x}}{x} \geq \frac{2}{x}.$$

As $\int_{1}^{\infty} \frac{2}{x} dx$ diverges and $\int_{1}^{\infty} \frac{2 + e^{-x}}{x} dx \geq \int_{1}^{\infty} \frac{2}{x} dx$, the integral we are interested in diverges also.

54. To begin this calculation we will focus on the indefinite integral:

$$\int v^3 \cdot e^{-Mv^2/(2RT)} dv.$$

To find the antiderivative we will use the substitution $w = \frac{Mv^2}{2RT}$ so that $v^2 = \frac{2RTw}{M}$ and $dv = \frac{RT}{M} \cdot \frac{dw}{v}$. Carrying out the substitution gives:

$$\int v^3 \cdot e^{-Mv^2/(2RT)} dv = 2\left(\frac{RT}{M}\right)^2 \cdot \int w \cdot e^{-w} \cdot dw.$$

Using integration by parts to evaluate the indefinite integral gives:

$$2\left(\frac{RT}{M}\right)^2 \cdot \int w \cdot e^{-w} \cdot dw = 2\left(\frac{RT}{M}\right)^2 \cdot \left(-w \cdot e^{-w} - e^{-w}\right) + C.$$

Substituting this result into the formula provided for $\bar{v}$ and simplifying gives the following.

$$\bar{v} = \frac{4}{\sqrt{\pi}} \cdot \left(\frac{M}{2RT}\right)^{\frac{3}{2}} \cdot 2 \cdot \left(\frac{RT}{M}\right)^{2} \cdot \mathrm{Lim}_{a\to\infty}\left[-w \cdot e^{-w} - e^{-w}\right]_{0}^{\frac{Ma^2}{2RT}} = \sqrt{\frac{8RT}{\pi M}} \cdot \mathrm{Lim}_{a\to\infty}\left[-w \cdot e^{-w} - e^{-w}\right]_{0}^{\frac{Ma^2}{2RT}}$$

Taking the limit as $a \to \infty$ gives:

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}} \cdot \mathrm{Lim}_{a\to\infty}\left[-w \cdot e^{-w} - e^{-w}\right]_{0}^{\frac{Ma^2}{2RT}} = \sqrt{\frac{8RT}{\pi M}}.$$