Solutions to Homework #2

Problems from Pages 319-320 (Section 6.2)

2. Here we will rewrite $\cos^2(x)$ as $1 - \sin^2(x)$ to put express the integrand as a polynomial in $\sin(x)$ with a single factor of $\cos(x)$ remaining.

$$\int \sin^6(x)\cos^3(x)dx = \int \sin^6(x)\cdot\left(1-\sin^2(x)\right)\cdot\cos(x)dx.$$

Now we will make the substitution $u = \sin(x)$ to integrate.

$$\int \sin^6(x)\cdot\left(1-\sin^2(x)\right)\cdot\cos(x)dx = \int\left(u^6-u^8\right)du = \tfrac{1}{7}u^7 - \tfrac{1}{9}u^9 + C = \tfrac{1}{7}\sin^7(x) - \tfrac{1}{9}\sin^9(x) + C$$

6. We will use the identity $\sin^2(mx) + \cos^2(mx) = 1$ to write the integral as a polynomial in cosine times a single remaining factor of sine.

$$\int \sin^3(mx)dx = \int\left(1-\cos^2(mx)\right)\sin(mx)dx.$$

Now we will make the substitution $u = \cos(mx)$ to integrate.

$$\int\left(1-\cos^2(mx)\right)\sin(mx)dx = \tfrac{-1}{m}\int\left(1-u^2\right)du = \tfrac{-1}{m}\left(u-\tfrac{1}{3}u^3\right)+C = \tfrac{-1}{m}\cos(mx)+\tfrac{1}{3m}\cos^3(mx)+C$$

20. We will use the identity $\tan^2(x) + 1 = \sec^2(x)$ to rewrite one of the factors of $\tan^2(x)$ that occur in the integral.

$$\int \tan^4(x)dx = \int \tan^2(x)\cdot\tan^2(x)dx = \int \tan^2(x)\cdot\left(\sec^2(x)-1\right)dx = \int \tan^2(x)\sec^2(x)dx - \int \tan^2(x)dx$$

The first integral can be evaluated using u-substitution and $u = \tan(x)$. Doing this gives:

$$\int \tan^2(x)\sec^2(x)dx = \int u^2du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3(x) + C.$$

To evaluate the second integral we can use the identity $\tan^2(x) + 1 = \sec^2(x)$ a second time. Doing this gives:

$$\int \tan^2(x)dx = \int\left(\sec^2(x)-1\right)dx = \tan(x) - x + C.$$

Putting all of these results together gives the solution to this problem.

$$\int \tan^4(x)dx = \tfrac{1}{3}\tan^3(x) - \tan(x) + x + C.$$

36.(a) To solve Part (a) we can make use of the angle addition formulas for sine:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b).$$

Adding these two formulas and dividing by two gives:

$$\tfrac{1}{2}\left[\sin(a+b) + \sin(a-b)\right] = \tfrac{1}{2}\left[\sin(a)\cos(b) + \sin(a)\cos(b)\right] = \sin(a)\cos(b).$$

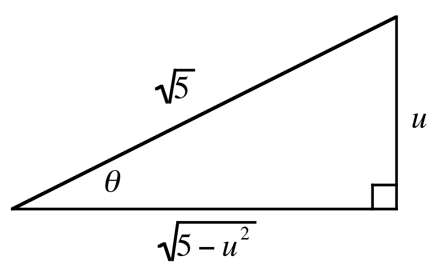
36.(b) Using the result from Part (a) to rewrite the integral here gives:

$$\int \sin(3x)\cos(x)dx = \int \tfrac{1}{2}\left[\sin(3x+x) + \sin(3x-x)\right]dx = \tfrac{-1}{8}\cos(4x) - \tfrac{1}{4}\cos(2x) + C.$$

50. Here we will make the trigonometric substitution $u = \sqrt{5}\cdot\sin(\theta)$ so that $du = \sqrt{5}\cdot\cos(\theta)\cdot d\theta$. Using this substitution the integral becomes:

$$\int \frac{1}{u\cdot\sqrt{5-u^2}}du = \int \frac{\sqrt{5}\cdot\cos(\theta)}{\sqrt{5}\cdot\sin(\theta)\cdot\sqrt{5}\cdot\cos(\theta)}d\theta = \frac{1}{\sqrt{5}}\int \csc(\theta)d\theta = \frac{1}{\sqrt{5}}\ln\left(\left|\csc(\theta) - \cot(\theta)\right|\right) + C$$

To convert the antiderivative back to the original variable (u) we will use the right angle triangle,



so that $\csc(\theta) = \frac{\sqrt{5}}{u}$ and $\cot(\theta) = \frac{\sqrt{5-u^2}}{u}$, and the final answer can be written:

$$\int \frac{1}{u\cdot\sqrt{5-u^2}}du = \frac{1}{\sqrt{5}}\ln\left(\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right|\right) + C.$$

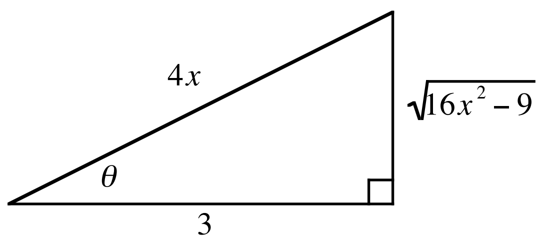
52. This integral can be matched to one of the three patterns that we have if we first factor out the factor of 16. Doing this gives:

$$\int \frac{dx}{x^2\sqrt{16x^2-9}}dx = \frac{1}{4}\int \frac{dx}{x^2\sqrt{x^2-\left(\frac{3}{4}\right)^2}}.$$

From here we make the trigonometric substitution $x = \frac{3}{4}\sec(\theta)$ so that $dx = \frac{3}{4}\sec(\theta)\tan(\theta)d\theta$. Using these, the integral can be transformed:

$$\int \frac{dx}{x^2\sqrt{16x^2-9}}dx = \frac{1}{4}\int \frac{\frac{3}{4}\sec(\theta)\tan(\theta)}{\left(\frac{3}{4}\right)^3\sec^2(\theta)\tan(\theta)}d\theta = \frac{4}{9}\int \cos(\theta)d\theta = \frac{4}{9}\sin(\theta) + C.$$

To convert back to the original variable (x) we can use the following right angle triangle to see that $\sin(\theta) = \frac{\sqrt{16x^2-9}}{4x}$.



This gives the final answer to this problem:

$$\int \frac{dx}{x^2\sqrt{16x^2-9}}dx = \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C.$$

54. In this problem you have to evaluate a definite integral. To do this, we will first find an antiderivative by forgetting about the limits of integration. When the antiderivative has been found, we will then go back and put the limits of integration back into evaluate the definite integral.

To evaluate the indefinite integral we will make the trigonometric substitution x = tan(θ) so that $dx = \sec^2(\theta)d\theta$. Then the indefinite integral becomes:

$$\int \sqrt{1+x^2}dx = \int \sec(\theta)\cdot\sec^2(\theta)\cdot d\theta.$$

To integrate we will use integration by parts with u = sec(θ) and $v' = \sec^2(\theta)$. Carrying this out gives:

$$\int \sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) - \int \sec(\theta)\cdot\tan^2(\theta)d\theta.$$

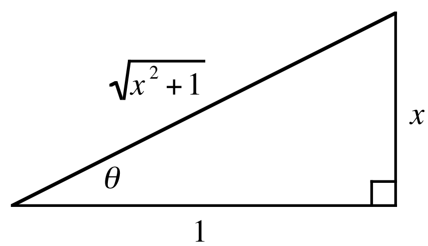
To evaluate this new integral, we will use the identity $\tan^2(x) + 1 = \sec^2(x)$ to rewrite the new integral. Doing this and then distributing gives the following:

$$\int \sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) - \int \sec^3(\theta)d\theta + \int \sec(\theta)d\theta.$$

Adding the integral of $\sec^3(\theta)$ to both sides, dividing by 2 and using the formula for integrating $\sec(\theta)$ gives the following antiderivative:

$$\int \sec^3(\theta)d\theta = \tfrac{1}{2}\sec(\theta)\tan(\theta) + \tfrac{1}{2}\ln\left(\left|\sec(\theta) + \tan(\theta)\right|\right) + C.$$

We can use the following right angle triangle to find expressions for $\sec(\theta)$ and $\tan(\theta)$ to convert the antiderivative back to the original variable, x.



From this diagram, $\tan(\theta) = x$ and $\sec(\theta) = \sqrt{x^2+1}$, so that the final answer to the problem may be computed:

$$\int_0^1 \sqrt{1+x^2}\,dx = \left[\tfrac{1}{2}x\sqrt{x^2+1} + \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^1 = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right].$$

Problems from Pages 327-328 (Section 6.3)

12. The denominator of the rational function factors into a pair of distinct linear factors:

$$x^2 + 3x + 2 = (x+1)(x+2),$$

so the partial fraction decomposition for the ration function will resemble:

$$\frac{x-1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Adding the two fractions and just paying attention to the numerators gives the equation:

$$x - 1 = A(x+2) + B(x+1),$$

which is rendered into the following two equations by equating coefficients of powers of x:

$$A + B = 1 \qquad \text{and} \qquad 2A + B = -1.$$

Solving these two equations gives $A = -2$ and $B = 3$ so that the integral can be evaluated in the following fashion:

$$\int \frac{x-1}{x^2+3x+2}\,dx = \int \frac{-2}{x+1}\,dx + \int \frac{3}{x+2}\,dx = -2 \cdot \ln(|x+1|) + 3 \cdot \ln(|x+2|) + C$$

18. The denominator of the rational function factors into three distinct linear factors:

$$x^3 - x = x \cdot \left(x^2 - 1\right) = x \cdot (x-1) \cdot (x+1).$$

The partial fraction decomposition for the rational function will resemble:

$$\frac{x^2+2x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

Adding the three fractions and just paying attention to the numerators gives the equation:

$$x^2 + 2x - 1 = A\left(x^2 - 1\right) + B\left(x^2 - x\right) + C\left(x^2 + x\right),$$

which is rendered into the following three equations by equating coefficients of powers of x:

$$A + B + C = 1 \qquad -B + C = 2 \qquad \text{and} \qquad -A = -1.$$

Solving these two equations gives $A = 1$, $B = -1$ and $C = 1$ so that the integral can be evaluated in the following fashion:

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-1}{x+1}\,dx + \int \frac{1}{x-1}\,dx = \ln(|x|) - \ln(|x+1|) + \ln(|x-1|) + C.$$

24. The denominator of the rational function is already factored, consisting of a repeated linear factor and an irreducible quadratic factor. The partial fractions decomposition for the rational function will therefore resemble:

$$\frac{x^2-2x-1}{(x-1)^2 \cdot \left(x^2+1\right)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Adding the three fractions together and paying attention to the numerators gives the following equation:

$$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2.$$

Note that since the denominator of the rational function does not include a factor of $(x - 1)^3$, we do not simply multiply each numerator by the denominators it lacks. Doing this would result in an additional, erroneous factor of $(x - 1)$ in the above equation.

FOILing the products and simplifying will give the following equation.

$$x^2 - 2x - 1 = A(x^3 - x^2 + x - 1) + B(x^2+1) + C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1).$$

Equating coefficients of powers of x will give four equations to determine A, B, C and D. These are:

$$\begin{array}{lllllll} A & & +C & & = & 0 \\ -A & +B & -2C & +D & = & 1 \\ A & & +C & -2D & = & -2 \\ -A & +B & & +D & = & -1 \end{array}$$

Solving these equations gives: $A = 1$, $B = -1$, $C = -1$ and $D = 1$, so that the integral can be evaluated as follows:

$$\int \frac{x^2 - 2x - 1}{(x-1)^2 \cdot (x^2+1)} dx = \int \frac{1}{x-1} dx + \int \frac{-1}{(x-1)^2} dx + \int \frac{-x}{x^2+1} dx + \int \frac{1}{x^2+1} dx,$$

so that with a little help from u-substitution we can obtain:

$$\int \frac{x^2 - 2x - 1}{(x-1)^2 \cdot (x^2+1)} dx = \ln(|x-1|) + (x-1)^{-1} - \tfrac{1}{2}\ln(|x^2+1|) + \arctan(x) + C.$$