Solutions to Homework #1

Problems from Pages 299-300 (Section 5.5)

14. Let $u = x^2 + 1$ so that $du = 2xdx$ and the indefinite integral can be evaluated via substitution as follows.

$$\int \tfrac{x}{(x^2+1)^2}\,dx = \int u^{-2}\tfrac{1}{2}\,du = \tfrac{-1}{2}u^{-1} + C = \frac{-1}{2(x^2+1)} + C.$$

18. Let $u = 2y^4 - 1$. Then $du = 8y^3dx$ and the indefinite integral can be evaluated via substitution as follows.

$$\int y^3\sqrt{2y^4-1}\,dy = \int u^{\frac{1}{2}}\tfrac{1}{8}\,du = \tfrac{1}{12}u^{\frac{3}{2}} + C = \tfrac{1}{12}\left(2y^4-1\right)^{\frac{3}{2}} + C.$$

34. This particular integral is a little deceptive and requires some coaxing to reveal its true nature. A clue to the substitution that will work well here ($u = x^2$ with $du = 2xdx$ is very effective) is the fact that there is only a single power of x to cancel with whatever power of x the du brings with it. A du that brings a single power of x (like $du = 2xdx$) will be perfect for the job.

$$\int \tfrac{x}{1+x^2}\,dx = \int \tfrac{1}{1+u^2}\tfrac{1}{2}\,du = \tfrac{1}{2}\tan^{-1}(u) + C = \tfrac{1}{2}\tan^{-1}\left(x^2\right) + C.$$

46. Again, this integral requires some study before the most helpful substitution is revealed. In this case, as x and u are linearly related ($u = 1 + 2x$ and $du = 2dx$), it is possible to solve for the x that remains in the integral after dx is replaced. In this case, we will replace x by $\tfrac{1}{2}(u-1)$.

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \int_1^9 \frac{x}{\sqrt{u}}\tfrac{1}{2}\,du = \frac{1}{2}\int_1^9 \frac{\tfrac{1}{2}(u-1)}{\sqrt{u}}du = \frac{1}{4}\int_1^9 u^{\frac{1}{2}} - u^{\frac{-1}{2}}\,du = \frac{1}{4}\left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9 = \frac{10}{3}.$$

60. The number of calculators will be equal to the definite integral of the rate of production from time 2 to time 4. This is equal to the following:

$$\int_2^4 5000\left(1 - \tfrac{100}{(t+10)^2}\right)dt = 5000\left[t + \tfrac{100}{(t+10)}\right]_2^4 \approx 4048 \text{ calculators.}$$

62. Here we let $u = x^2$ so that $du = 2xdx$ and the substitution proceeds as follows.

$$\int_0^3 xf\left(x^2\right)dx = \frac{1}{2}\int_0^9 f(u)du = \tfrac{1}{2}(4) = 2.$$

Problems from Pages 309-310 (Section 6.1)

8. Let $u = x^2$ and $v' = \cos(mx)$. Then $u' = 2x$ and $v = \frac{1}{m}\sin(mx)$. Plugging these expressions into the integration by parts formula gives:

$$\int x^2\cos(mx)dx = \tfrac{1}{m}x^2\sin(mx) - \tfrac{2}{m}\int x\sin(mx)dx.$$

This new integral must be integrated by parts with $u = x$ and $v' = \sin(mx)$. Then $u' = 1$ and $v = \frac{-1}{m}\cos(mx)$. Once again plugging into the integration by parts formula gives:

$$\int x\sin(mx)dx = \tfrac{-1}{m}x\sin(mx) + \tfrac{1}{m}\int\cos(mx)dx = \tfrac{-1}{m}x\sin(mx) + \tfrac{1}{m^2}\sin(mx) + C.$$

Substituting this into the first integration formula solves the problem.

$$\int x^2\cos(mx)dx = \tfrac{1}{m}x^2\sin(mx) + \tfrac{2}{m^2}x\cos(mx) - \tfrac{2}{m^3}\sin(mx) + C.$$

16. Let $u = x^2 + 1$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$. Plugging these into the integration by parts formula gives:

$$\int\left(x^2+1\right)e^{-x}dx = -\left(x^2+1\right)e^{-x} + 2\int xe^{-x}dx.$$

To evaluate this new integral we must integrate by parts a second time. Now let $u = x$ and again set $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Plugging these into the integration by parts formula gives:

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C.$$

Substituting this into the first integral formula gives the antiderivative.

$$\int\left(x^2+1\right)e^{-x}dx = -\left(x^2+1\right)e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

To evaluate the definite integral, use the above antiderivative formula and the Fundamental Theorem of Calculus.

$$\int_0^1 \left(x^2+1\right)e^{-x}dx = \left[-\left(x^2+1\right)e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1 = -6e^{-1}+3.$$

22. This problem can be solved by parts, u-substitution (or trigonometric substitution). The solution given here uses u-substitution with $u = 4 + r^2$. Then:

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr = \frac{1}{2}\int_4^5 \frac{u-4}{\sqrt{u}}du = \frac{1}{2}\int_4^5 u^{\frac{1}{2}} - 4u^{\frac{-1}{2}}du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^5 = \frac{16}{3} - \frac{7\sqrt{5}}{3}.$$

34. Let $u = x^n$ and $v' = e^x$. Then $u' = nx^{n-1}$ and $v = e^x$. Plugging these into the integration by parts formula gives the desired reduction formula.

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx.$$